

# SPECTRAL CHARACTERS OF INTEGRABLE REPRESENTATIONS OF TOROIDAL LIE ALGEBRAS

TANUSREE KHANDAI

**ABSTRACT.** We study the category of integrable representations with finite-dimensional weight spaces of toroidal Lie algebras on which the centre acts non-trivially. We show that any indecomposable object in this category has a composition series and every module in this category can be written as the direct sum of finitely many indecomposable modules. Finally we give a parametrization of the blocks in this category.

## 1. INTRODUCTION

A  $k$ -toroidal Lie algebra  $\mathcal{T}(\mathfrak{g})$  associated to a finite-dimensional complex simple Lie algebra  $\mathfrak{g}_{fin}$  is the universal central extension of the Lie algebra of polynomial maps from  $(\mathbb{C}^*)^k$  to  $\mathfrak{g}_{fin}$ . They are a natural generalization of the affine Kac-Moody Lie algebras  $\mathfrak{g}_{aff}$  whose representation theory has been a subject of interest for the past three decades [C, CG, CM, CP1, CP2, Kac, R2, VV]. In contrast to the one-dimensional center of  $\mathfrak{g}_{aff}$ , the  $k$ -toroidal Lie algebras have a  $\mathbb{Z}^k$ -graded infinite-dimensional center and this makes the study of the integrable representations of these Lie algebras more interesting and involved. In recent years the category of integrable representations of toroidal Lie algebras and its quotients by central ideals of finite co-dimension have been studied in a number of papers [CFK, CL, FL, Kh, KL, NS, S, R1, R3, RSF] and the simple objects of the category  $\mathcal{I}_{fin}$  of integrable  $\mathcal{T}(\mathfrak{g})$ -modules with finite-dimensional weight spaces have been classified in [R1, Kh]. In this paper we continue with our study and determine the structure of the full subcategory  $\mathcal{I}_{fin}^*$  of  $\mathcal{I}_{fin}$  consisting of  $\mathcal{T}(\mathfrak{g})$ -modules on which the center acts non-trivially.

In [CM] it was shown that the blocks in the category  $\mathcal{F}$  of finite-dimensional representations of  $\mathfrak{g}_{aff}$  can be associated with finitely supported functions from  $\mathbb{C}^*$  to  $\Gamma$ , the quotient of the weight lattice of  $\mathfrak{g}_{fin}$  by the root lattice of  $\mathfrak{g}_{fin}$ . It was further shown that such an association is unique upto a scaling. With an understanding of the structure of  $\mathcal{F}$  it was proved in [CG] that the category of graded level zero integrable representations with finite-dimensional weight spaces of  $\mathfrak{g}_{aff}$  has a block decomposition and blocks in this category are parametrized by orbits for a natural action of the group  $\mathbb{C}^*$  on the set of finitely supported functions from  $\mathbb{C}^*$  to  $\Gamma$ . The first extension groups for finite-dimensional irreducible representations of generalized current Lie algebras and equivariant map algebras which include the multiloop Lie algebras was studied in [Ko] and [NS] respectively. Mimicking the methods in [CG] and using the results from [NS] it can be easily checked that the full subcategory  $\mathcal{I}_{fin}^{(0)}$  of  $\mathcal{I}_{fin}$  consisting of  $\mathcal{T}(\mathfrak{g})$ -modules on which the central elements act trivially, has a block decomposition and upto a scaling factor, finitely supported functions from  $(\mathbb{C}^*)^k$  to  $\Gamma$  parametrize the blocks in  $\mathcal{I}_{fin}^{(0)}$ . While the structure of the category  $\mathcal{I}_{fin}^{(0)}$  is similar to the category of integrable level zero  $\mathfrak{g}_{aff}$ -modules with finite-dimensional weight spaces, relatively little is known about the structure of  $\mathcal{I}_{fin}^*$ . Following an approach streamlined with [CG] we study the category  $\mathcal{I}_{fin}^*$  in this paper.

We now explain the results in some details. Let  $K_1, K_2, \dots, K_k$  be the elements of the  $k$ -toroidal Lie algebra that span the space of zero-graded central elements of  $\mathcal{T}(\mathfrak{g})$ . Given an integrable indecomposable  $\mathcal{T}(\mathfrak{g})$ -module  $V$  in  $\mathcal{I}_{fin}^*$ , by standard theory there exists integers  $m_1, \dots, m_k$  such that  $K_i.v = m_i v$  for all  $v \in V$ . Let  $\mathcal{I}_{fin}^{(\mathbf{m})}$  be the full subcategory of  $\mathcal{I}_{fin}^*$  on which the zero graded central element  $K_i$  acts by the integer  $m_i$  for  $i = 1, 2, \dots, k$ . It has been proved in [R2, Kh] that given  $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k - \mathbf{0}$  the category  $\mathcal{I}_{fin}^{(\mathbf{m})}$  is equivalent to  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  where  $m = \gcd(m_1, \dots, m_k)$ ,  $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{Z}^k$  and  $m\mathbf{e}_1 = (m, 0, \dots, 0)$ . Thus to study the structure of  $\mathcal{I}_{fin}^*$  it suffices to study the structure of the category  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  for a non-zero integer  $m$ . In Section 2 we set the notation for the paper and recall preliminary results on toroidal Lie algebras and their representation theory. Using the semi-simplicity of the category of integrable  $\mathfrak{g}_{aff}$ -modules of non-zero level with finite dimensional weight spaces we show in Section 3 that every indecomposable object in the category  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  is both Artinian and Noetherian and hence has a composition series. Further it is shown that  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  has a block decomposition, that is, every object in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  can be written as a direct sum of finitely many indecomposables.

In Section 5 we give a parametrization of the blocks in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$ . It is observed that in the case when  $\mathfrak{g}_{fin}$  the finite-dimensional Lie algebra associated to the Lie algebra  $\mathcal{T}(\mathfrak{g})$  is non-simply laced or when  $\Gamma$ , the quotient group of the weight lattice by root lattice of  $\mathfrak{g}_{fin}$  is trivial, the blocks in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  are parametrized by finitely supported functions from  $(\mathbb{C}^*)^{k-1}$  to the additive group  $\mathbb{Z} \times \Gamma$ , that are unique upto a scaling. In the other cases however such a straight-forward parametrization is not possible. In order to deal with the anomalies we introduce the notion of finitely supported functions from  $(\mathbb{C}^*)^{k-1}$  to  $\mathbb{Z} \times \Gamma$  of type **I** and of type **II**. We then show that while upto a scaling the functions of type **I** correspond to unique blocks in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$ , the subcategory associated with a function of type **II** decomposes into blocks that are determined by isomorphism classes of  $\mathcal{T}(\mathfrak{g})$ -modules which were determined in [Kh].

## 2. PRELIMINARIES

In this section we fix the notations for the paper and recall the explicit realization of  $k$ -toroidal Lie algebras from [R2, RM].

**2.1.** Throughout the paper  $\mathbb{C}, \mathbb{R}, \mathbb{Z}$  and  $\mathbb{N}$  shall denote the field of complex numbers, real numbers, the set of integers and the set of natural numbers. For a commutative associative algebra  $\mathbf{A}$ , the set of maximal ideals of  $\mathbf{A}$  shall be denoted by  $\max \mathbf{A}$  and for a Lie algebra  $\mathfrak{a}$  the universal enveloping algebra of  $\mathfrak{a}$  shall be denoted by  $\mathcal{U}(\mathfrak{a})$ . For  $k \in \mathbb{N}$ , a  $k$ -tuple of integers  $(m_1, \dots, m_k)$  shall be denoted by  $\mathbf{m}$  and given  $\mathbf{m} \in \mathbb{Z}^k$ ,  $\underline{\mathbf{m}}$  shall denote the  $(k-1)$ -tuple of integers  $(m_2, \dots, m_k)$ .

**2.2.** Let  $\mathfrak{g}_{fin}$  be a finite-dimensional simple Lie algebra of rank  $n$ ,  $\mathfrak{h}_{fin}$  a Cartan subalgebra of  $\mathfrak{g}_{fin}$  and  $R_{fin}$  the root system of  $\mathfrak{g}_{fin}$  with respect to  $\mathfrak{h}_{fin}$ . Let  $\{\alpha_i : 1 \leq i \leq n\}$  (respectively  $\{\omega_i : 1 \leq i \leq n\}$ ) be a set of simple roots (respectively fundamental weights) of  $\mathfrak{g}_{fin}$  with respect to  $\mathfrak{h}_{fin}$ ,  $R_{fin}^+$  (respectively  $Q_{fin}, P_{fin}$ ) be the corresponding set of positive roots (respectively root lattice and weight lattice) and let  $\theta$  (respectively  $\theta_s$ ) be the highest root (highest short root) of  $\mathfrak{g}_{fin}$  if  $\mathfrak{g}$  is simply-laced (respectively not simply-laced). Let  $Q_{fin}^+$  and  $P_{fin}^+$  be the  $\mathbb{Z}_+$  span of the simple roots and fundamental weights of  $(\mathfrak{g}_{fin}, \mathfrak{h}_{fin})$ .

Given  $\alpha \in R_{fin}^\pm$  let  $\mathfrak{g}_{fin}^{\pm\alpha}$  denote the corresponding root space. Let  $x_\alpha^\pm \in \mathfrak{g}_{fin}^{\pm\alpha}$  and  $\alpha^\vee \in \mathfrak{h}_{fin}$  be fixed elements such that  $\alpha^\vee = [x_\alpha^+, x_\alpha^-]$  and  $[\alpha^\vee, x_\alpha^\pm] = \pm 2x_\alpha^\pm$ .

Let  $\Gamma = P_{fin}/Q_{fin}$ . It is well known that  $\Gamma$  is a finite abelian group and the non-zero elements in  $\Gamma$  are of the form  $\omega_i \bmod Q_{fin}$  where  $\omega_i$  is a fundamental weight of  $\mathfrak{g}_{fin}$  such that  $\omega_i(\alpha) = 0$  or 1 for all  $\alpha \in R_{fin}^+$ . Let

$$J_0 = \begin{cases} \{1, 2, \dots, n\} & \text{if } \mathfrak{g} \text{ is of type } A_n \\ \{n\} & \text{if } \mathfrak{g} \text{ is of type } B_n \\ \{1\} & \text{if } \mathfrak{g} \text{ is of type } C_n \\ \{1, n-1, n\} & \text{if } \mathfrak{g} \text{ is of type } D_n \\ \{1, 6\} & \text{if } \mathfrak{g} \text{ is of type } E_6 \\ \{7\} & \text{if } \mathfrak{g} \text{ is of type } E_7 \end{cases}$$

By [H, Section 13, Exercise 13], it is known that for  $\mathfrak{g}_{fin}$  of type  $A_n, B_n, C_n, D_n, E_6, E_7$ ,

$$\Gamma = \{\omega_i \bmod Q_{fin} : i \in J_0\}$$

and for  $\mathfrak{g}_{fin}$  of type  $E_8, F_4, G_2$ ,  $\Gamma = \{0\}$ .

For  $\lambda \in P_{fin}^+$ , let  $V(\lambda)$  denote the cyclic  $\mathfrak{g}_{fin}$ -module generated by a weight vector  $v_\lambda$  with defining relations:

$$x_\alpha^+ \cdot v_\lambda = 0, \quad \forall \alpha \in R_{fin}^+, \quad h \cdot v_\lambda = \lambda(h)v_\lambda, \quad \forall h \in \mathfrak{h}_{fin}, \quad (x_\alpha^-)^{\lambda(\alpha)+1} \cdot v_\lambda = 0, \quad \forall \alpha \in R_{fin}^+.$$

It is well known that  $V(\lambda)$  is an irreducible finite-dimensional  $\mathfrak{g}_{fin}$ -module with highest weight  $\lambda$  and any irreducible finite-dimensional  $\mathfrak{g}_{fin}$ -module is isomorphic to  $V(\mu)$  for  $\mu \in P_{fin}^+$ .

**2.3.** For a positive integer  $k$ , let  $\mathbb{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$  be the Laurent polynomial ring in  $k$  commuting variables  $t_1, \dots, t_k$  and for  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{Z}^k$  define  $t^{\mathbf{r}} := t_1^{r_1} t_2^{r_2} \dots t_k^{r_k}$ . Let

$$L_k(\mathfrak{g}) = \mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}]$$

be an infinite-dimensional vector space and let  $\mathcal{Z} = \Omega_k/dL_k$  be the module of differentials of  $\mathbb{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$  spanned by the set of vectors  $\{t^{\mathbf{m}} K_i, \mathbf{m} \in \mathbb{Z}^k, 1 \leq i \leq k\}$  together with the relations  $\sum_{i=1}^k r_i t^{\mathbf{r}} K_i = 0$ , for all  $\mathbf{r} \in \mathbb{Z}^k$ . By [K] the universal central extension of  $L_k(\mathfrak{g})$  is the Lie algebra  $\tilde{L}_k(\mathfrak{g}) := L_k(\mathfrak{g}) \oplus \mathcal{Z}$ , with Lie bracket given by

$$[x \otimes P, y \otimes Q] = [x, y] \otimes PQ + \overline{Q(dP)}(x|y), \quad [x \otimes P, \omega] = 0 \quad [\omega, \omega'] = 0, \quad (2.1)$$

where  $x, y \in \mathfrak{g}$ ,  $P, Q \in L_k$ ,  $\omega, \omega' \in \mathcal{Z}$  and  $\overline{Q(dP)}$  is the residue class of  $QdP$  in  $\mathcal{Z}$ . Let  $d_i : \tilde{L}_k(\mathfrak{g}) \rightarrow \tilde{L}_k(\mathfrak{g})$ ,  $1 \leq i \leq k$  be the  $k$  derivations obtained by extending linearly the assignment

$$d_i(x \otimes t^{\mathbf{m}}) = m_i x \otimes t^{\mathbf{m}}, \quad d_i(t^{\mathbf{m}} K_j) = m_i t^{\mathbf{m}} K_j \quad \forall \quad 1 \leq i, j \leq k. \quad (2.2)$$

Let  $D_k$  be the  $\mathbb{C}$  linear span of the derivations  $d_1, d_2, \dots, d_k$ . The  $k$ -toroidal Lie algebra  $\mathcal{T}(\mathfrak{g})$  associated to a simple Lie algebra  $\mathfrak{g}_{fin}$  is obtained by adjoining to the universal central extension  $\tilde{L}_k(\mathfrak{g})$  of  $L_k(\mathfrak{g})$  the space of derivations  $D_k$ . Explicitly,

$$\mathcal{T}(\mathfrak{g}) := L_k(\mathfrak{g}) \oplus \mathcal{Z} \oplus D_k$$

with the Lie bracket given by (2.1) and (2.2). Let  $\mathfrak{h}_{tor}$  be the subalgebra of  $\mathcal{T}(\mathfrak{g})$  given by

$$\mathfrak{h}_{tor} := \mathfrak{h}_{fin} \oplus \mathcal{Z} \oplus D_k.$$

In order to identify  $\mathfrak{h}_{fin}^*$  with a subspace of  $\mathfrak{h}_{tor}^*$ , an element  $\lambda \in \mathfrak{h}_{fin}^*$  is extended to an element of  $\mathfrak{h}_{tor}^*$  by setting  $\lambda(c) = 0 = \lambda(d_i) = 0$ , for all  $c \in \mathcal{Z}$ ,  $(1 \leq i \leq k)$ . For  $1 \leq i \leq k$ , define  $\delta_i \in \mathfrak{h}_{tor}^*$  by

$$\delta_i|_{\mathfrak{h}_{fin} + \mathcal{Z}} = 0, \quad \delta_i(d_j) = \delta_{ij}, \quad \text{for } 1 \leq j \leq k.$$

Given  $\alpha \in R_{fin}$  and  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^k$ , set  $\alpha + \delta_{\mathbf{m}} = \alpha + \sum_{i=1}^k m_i \delta_i$  and let

$$R_{tor}^e = \{\alpha + \delta_{\mathbf{m}} : \alpha \in R_{fin}, \mathbf{m} \in \mathbb{Z}^k\} \quad \text{and} \quad R_{tor}^{im} = \{\delta_{\mathbf{m}} = \sum_{i=1}^k m_i \delta_i : \mathbf{m} \in \mathbb{Z}^k - \{\mathbf{0}\}\}.$$

The sets  $R_{tor}^e$  and  $R_{tor}^{im}$  are respectively called the real and imaginary roots of  $\mathcal{T}(\mathfrak{g})$  and the subset  $R_{tor} := R_{tor}^e \cup R_{tor}^{im}$  of  $\mathfrak{h}_{tor}^*$  is called the set of roots of  $\mathcal{T}(\mathfrak{g})$  with respect to  $\mathfrak{h}_{tor}$ . Setting

$$\alpha_{n+i} := \delta_i - \theta, \quad \text{for } i = 1, \dots, k,$$

it can be seen that the set  $\Delta_{tor} = \{\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}\}$  forms a simple system for  $R_{tor}$ . For  $\gamma \in R_{tor}$ , let  $\mathcal{T}_{\gamma}$  denote the corresponding root space. It is observed that

$$\mathcal{T}_{\alpha+\delta_{\mathbf{m}}} = t^{\mathbf{m}} \otimes \mathfrak{g}_{fin}^{\alpha}, \quad \text{for } \alpha + \delta_{\mathbf{m}} \in R_{tor}^e, \quad \text{and} \quad \mathcal{T}_{\delta_{\mathbf{m}}} = t^{\mathbf{m}} \otimes \mathfrak{h}_{fin}, \quad \text{for } \delta_{\mathbf{m}} \in R_{tor}^{im}.$$

Clearly,  $\mathcal{T}_{\alpha+\delta_{\mathbf{m}}}$  and  $\mathcal{T}_{\delta_{\mathbf{m}}}$  are  $\mathfrak{h}_{tor}$ -stable subspaces of  $\mathcal{T}(\mathfrak{g})$  and one has the root space decomposition

$$\mathcal{T}(\mathfrak{g}) = \mathfrak{h}_{tor} \oplus \left( \bigoplus_{\gamma \in R_{tor}} \mathcal{T}_{\gamma} \right).$$

Given  $\gamma = \alpha + \delta_{\mathbf{m}} \in R_{tor}^e$ , with  $\alpha \in R_{fin}^+$  and  $\mathbf{m} \in \mathbb{Z}^k$ , let  $\gamma^{\vee} = \alpha^{\vee} + \frac{2}{|\alpha|^2} \sum_i m_i K_i$ . With the given Lie bracket operation on  $\mathcal{T}(\mathfrak{g})$  it is easy to check that the subalgebra of  $\mathcal{T}(\mathfrak{g})$  spanned by  $\{x_{\alpha}^+ \otimes t^{\mathbf{m}}, x_{\alpha}^- \otimes t^{-\mathbf{m}}, \gamma^{\vee}\}$  is isomorphic to  $\mathfrak{sl}_2$  and we denote it by  $\mathfrak{sl}_2(\gamma)$ .

**2.4.** For  $k = 1$ , the Lie algebra  $\mathcal{T}(\mathfrak{g})$  is called an affine Kac-Moody Lie algebra and we denote it by  $\mathfrak{g}_{aff}$ . Explicitly,  $\mathfrak{g}_{aff} = \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}] \oplus \mathbb{C}K_1 \oplus \mathbb{C}d_1$ , where  $K_1$  is a and  $d_1$  is a derivation. Using the natural ordering on  $\mathbb{Z}$ , the set of real and imaginary roots of  $\mathfrak{g}_{aff}$  can be partitioned as follows:

$$R_{aff}^{e\pm 1} = \{\alpha + m\delta_1 : \alpha \in R_{fin}, m \in \mathbb{Z}_{\pm} \setminus \{0\}\} \cup R_{fin}^{\pm}, \quad R_{aff}^{im\pm} = \{m\delta_1 : m \in \mathbb{Z}_{\pm} \setminus \{0\}\}.$$

The set  $R_{aff}^+ = R_{aff}^{e+} \cup R_{aff}^{im+}$  (respectively  $R_{aff}^- = R_{aff}^{e-} \cup R_{aff}^{im-}$ ) is called the set of positive (respectively negative) roots of  $\mathfrak{g}_{aff}$  and  $R_{aff} = R_{aff}^+ \cup R_{aff}^-$  is the set of roots of  $\mathfrak{g}_{aff}$ . Denoting the root space of  $\mathfrak{g}_{aff}$  corresponding to a root  $\gamma \in R_{aff}$  by  $\mathfrak{g}_{aff}^{\gamma}$ , set

$$\mathfrak{n}_{aff}^{\pm} = \bigoplus_{\gamma \in R_{aff}^{\pm}} (\mathfrak{g}_{aff}^{\gamma}) \quad \text{and} \quad \mathfrak{h}_{aff} = \mathfrak{h}_{fin} \oplus \mathbb{C}K_1 \oplus \mathbb{C}d_1.$$

Then one has

$$\mathfrak{g}_{aff} = \mathfrak{n}_{aff}^- \oplus \mathfrak{h}_{aff} \oplus \mathfrak{n}_{aff}^+, \quad \text{and} \quad \mathfrak{h}_{aff}^* = \mathfrak{h}_{fin}^* \oplus \mathbb{C}\delta_1 \oplus \mathbb{C}\Lambda_{n+1},$$

where  $\Lambda_{n+1}, \delta_1 \in \mathfrak{h}_{aff}^*$  are such that  $\Lambda_{n+1}|_{\mathfrak{h}_{fin}^*} = 0 = \Lambda_{n+1}(d_1)$ ,  $\Lambda_{n+1}(K_1) = 1$  and  $\delta_1|_{\mathfrak{h}_{fin}^*} = 0 = \delta_1(K_1) = 0$ ,  $\delta_1(d) = 1$ . It thus follows that an element  $\lambda$  in  $\mathfrak{h}_{aff}^*$  can be uniquely written as

$$\lambda = \lambda(c)\Lambda_{n+1} + \lambda|_{\mathfrak{h}_{fin}^*} + \lambda(d_1)\delta_1,$$

where  $\lambda|_{\mathfrak{h}_{fin}^*}$  is the restriction of  $\lambda$  to  $\mathfrak{h}_{fin}^*$ . The set of simple roots  $\Delta_{aff}$  and coroots  $\Delta_{aff}^{\vee}$  of  $\mathfrak{g}_{aff}$  are respectively given by

$$\Delta_{aff} = \{\alpha_1, \dots, \alpha_n, \alpha_{n+1} = \delta_1 - \theta\}, \quad \text{and} \quad \Delta_{aff}^{\vee} = \{\alpha_1^{\vee}, \dots, \alpha_n^{\vee}, \alpha_{n+1}^{\vee} = K_1 - \theta^{\vee}\}.$$

Let  $Q_{aff}$  (respectively  $Q_{aff}^{\vee}$ ) be the root lattice (respectively coroot lattice) for  $\mathfrak{g}_{aff}$ . Clearly the central element  $K_1 \in Q_{aff}^{\vee}$  and there exists  $a_1^{\vee}, \dots, a_n^{\vee}, a_{n+1}^{\vee} \in \mathbb{Z}_+$  such that  $K_1 = \sum_{i=1}^{n+1} a_i^{\vee} \alpha_i^{\vee}$ . Let  $\Lambda_i$  ( $i = 1, \dots, n, n+1$ ) be the fundamental weights of  $\mathfrak{g}_{aff}$ , that is,  $\langle \Lambda_i, \alpha_j^{\vee} \rangle = \delta_{ij}$ , for  $1 \leq j \leq n+1$  and  $\Lambda_i(d) = 0$ . Then for  $1 \leq i \leq n$ ,  $\Lambda_i = a_i^{\vee} \Lambda_{n+1} + \omega_i$ , where  $\omega_1, \dots, \omega_n$  are the fundamental weights of

$\mathfrak{g}_{fin}$ . The set of integral weights of  $\mathfrak{g}_{aff}$ , which we denote by  $P_{aff}$ , is given by  $P_{aff} = \sum_{i=1}^{n+1} \mathbb{Z}\Lambda_i + \mathbb{C}\delta$ , and the set of dominant integral weights of  $\mathfrak{g}_{aff}$  is of the form  $P_{aff}^+ = \sum_{i=1}^{n+1} \mathbb{Z}_+\Lambda_i + \mathbb{C}\delta$ . Let  $\succeq$  be the partial order on  $P_{aff}$  defined by  $\lambda \succeq \mu$  if  $\lambda, \mu \in P_{aff}$  are such that  $\lambda - \mu \in \sum_{i=1}^{n+1} \mathbb{Z}_+\alpha_i$ . Given  $\lambda, \mu \in P_{aff}$  we shall write  $\lambda \succ \mu$  whenever  $\lambda \succeq \mu$  but  $\lambda \neq \mu$ .

A  $\mathfrak{h}_{aff}$ -diagonalizable module  $V$  of  $\mathfrak{g}_{aff}$  is said to be integrable if the root vectors corresponding to the real roots of  $\mathfrak{g}_{aff}$  are locally nilpotent on  $V$ . An integral  $\mathfrak{g}_{aff}$ -module is said to be of positive (respectively negative) level if the central element  $K_1$  acts on  $V$  by a positive (respectively negative) integer. Given  $\lambda \in P_{aff}^+$ , let  $X(\lambda)$  be the irreducible integrable  $\mathfrak{g}_{aff}$ -module with highest weight  $\lambda$  and highest weight vector  $v_\lambda$ . As  $X(\lambda)$  is integrable we can write it as a direct sum of its weight spaces :

$$X(\lambda) = \bigoplus_{\nu \in P_{aff}} X(\lambda)_\nu, \quad \text{where } X(\lambda)_\nu = \{v \in X(\lambda) : h.v = \nu(h)v, \forall h \in \mathfrak{h}_{aff}\}.$$

Let  $P_{aff}(\lambda) = \{\nu \in P_{aff} : X(\lambda)_\nu \neq 0\}$  and  $X^*(\lambda) := \bigoplus_{\nu \in P_{aff}} (X(\lambda)_\nu)^* \subset X(\lambda)^*$ . It was shown in [Kac] that  $X^*(\lambda)$  is a  $\mathfrak{g}_{aff}$ -submodule of  $X(\lambda)^*$  and is an irreducible integrable  $\mathfrak{g}_{aff}$ -module with lowest weight  $-\lambda$  and lowest weight vector  $v_\lambda^*$ , satisfying the relations:

$$\mathfrak{n}_{aff}^- . v_\lambda^* = 0, \quad h.v_\lambda^* = -\lambda(h)v_\lambda^*, \quad \forall h \in \mathfrak{h}_{aff}, \quad (x_{\alpha_i}^+)^{\lambda(\alpha_i^\vee)+1} . v_\lambda^* = 0, \quad \forall 1 \leq i \leq n+1.$$

We now record some results on integrable  $\mathfrak{g}_{aff}$ -modules with finite-dimensional weight spaces that we will need. Part(i) of the proposition was proved in [C], part(ii) was proved in [R1], part(iii) is a consequence of [R1, Lemma 1.6] and [CG, Lemma 2.1] and parts(iv-v) were proved in [CP2].

**Proposition.** *Let  $V$  be an integrable module with finite-dimensional weight spaces for  $\mathfrak{g}_{aff}$ .*

- i. *If  $V$  is an irreducible  $\mathfrak{g}_{aff}$ -module on which the center acts by a positive (respectively negative) integer then  $V$  is isomorphic to  $X(\Lambda)$  (respectively  $X^*(\Lambda)$ ) for some  $\Lambda \in P_{aff}^+$ .*
- ii. *If all eigenvalues of  $K_1$  are non-zero, then  $V$  is completely reducible as  $\mathfrak{g}_{aff}$ -module.*
- iii. *Let  $\lambda \in P_{aff}^+$  be of the form  $\lambda = \lambda(K_1)\Lambda_{n+1} + \lambda|_{\mathfrak{h}_{fin}} + \lambda(d_1)\delta_1$ . If  $\varpi_\lambda$  is the unique minimal element in  $P_{fin}^+$  such that  $\lambda|_{\mathfrak{h}_{fin}} \equiv \varpi_\lambda \pmod{Q_{fin}}$  then  $\varpi_{\lambda,r} = \lambda(K_1)\Lambda_{n+1} + \varpi_\lambda + r\delta_1 \in P_{aff}(\lambda)$  for all  $r \in \mathbb{C}$  such that  $\lambda(d) - r \in \mathbb{Z}_+$ . In particular, if  $\lambda|_{\mathfrak{h}_{fin}} \in P_{fin}^+ \cap Q_{fin}^+$  and  $\lambda|_{\mathfrak{h}_{fin}} \neq 0$ , then  $\lambda(K_1)\Lambda_{n+1} + \beta + r\delta_1 \in P_{aff}(\lambda)$  for all  $r \in \mathbb{C}$  such that  $\lambda(d_1) - r \in \mathbb{Z}_+$ , where  $\beta = \theta$  if  $\mathfrak{g}$  is simply-laced and  $\beta = \theta_s$  otherwise.*
- iv. *Let  $\Lambda, \mu \in P_{aff}^+$ . The  $\mathfrak{g}_{aff}$ -module  $X(\Lambda) \otimes X^*(\mu)$  is cyclic on  $v_\Lambda \otimes v_\mu^*$  with the relations*

$$\begin{aligned} (x_{\alpha_i}^-)^{\Lambda(\alpha_i^\vee)+1} . (v_\Lambda \otimes v_\mu^*) &= 0 = (x_{\alpha_i}^+)^{\mu(\alpha_i^\vee)+1} . (v_\Lambda \otimes v_\mu^*) \quad \forall 1 \leq i \leq n+1, \\ h.(v_\Lambda \otimes v_\mu^*) &= (\Lambda - \mu)(h)(v_\Lambda \otimes v_\mu^*), \quad \forall h \in \mathfrak{h}_{aff}. \end{aligned}$$

- v. *If  $\Lambda, \mu \in P_{aff}^+$  are such that  $\Lambda(\alpha_{n+1}^\vee) = m = \mu(\alpha_{n+1}^\vee)$  then there exists a non-zero  $\mathfrak{g}_{aff}$ -module homomorphism  $H : X(\Lambda) \otimes X^*(\mu) \rightarrow V(w((\Lambda - \mu)|_{\mathfrak{h}_{fin}})) \otimes \mathbb{C}[t_1^{\pm 1}]$ , where  $w$  is an element of the Weyl group  $W_{fin}$  of  $(\mathfrak{g}_{fin}, \mathfrak{h}_{fin})$  such that  $w((\Lambda - \mu)|_{\mathfrak{h}_{fin}}) \in P_{fin}^+$ .*

**Remark.** By [H, Chapter III. 13.4, Ex 13] if  $\lambda \in P_{fin}^+$ ,  $\lambda \notin Q_{fin}^+$  then there exists a unique  $i \in J_0$  such that  $\lambda \equiv \omega_i \pmod{Q_{fin}}$ . On the other hand by [Kac, Table Aff 1, Table Aff 2](also refer to diagram in Section 5.4),  $a_i^\vee = 1$  for each  $j \in J_0$ . Hence by Proposition 2.4(iii), if  $\lambda = \lambda(K_1)\Lambda_{n+1} + \lambda|_{\mathfrak{h}_{fin}} + \lambda(d_1)\delta_1 \in P_{aff}^+$  with  $\lambda|_{\mathfrak{h}_{fin}} \notin Q_{fin}^+$ , then there exists a unique  $i \in J_0$  such that  $(\lambda(K_1) - 1)\Lambda_{n+1} + \Lambda_i \in P_{aff}(\Lambda)$ , where the set  $J_0$  is as defined in section 2.2.

**2.5.** A  $\mathcal{T}(\mathfrak{g})$ -module is said to be integrable if it is  $\mathfrak{h}_{tor}$ -diagonalizable and the root vectors corresponding to the real roots of  $\mathcal{T}(\mathfrak{g})$  are locally nilpotent on  $V$ . Thus an integrable  $\mathcal{T}(\mathfrak{g})$ -module is of the form

$$V = \bigoplus_{\lambda \in \mathfrak{h}_{tor}^*} V_\lambda, \quad \text{where } V_\lambda = \{v \in V : hv = \lambda(h)v, \text{ for all } h \in \mathfrak{h}_{tor}\}.$$

We set  $P(V) = \{\lambda \in \mathfrak{h}_{tor}^* : V_\lambda \neq 0\}$  as the set of all weights of an integrable  $\mathcal{T}(\mathfrak{g})$ -module  $V$ .

Let  $\mathcal{I}$  be the category of integrable  $\mathcal{T}(\mathfrak{g})$ -modules and morphisms

$$\text{hom}_{\mathcal{I}}(V, V') = \text{hom}_{\mathcal{T}(\mathfrak{g})}(V, V'), \quad V, V' \in \text{Ob } \mathcal{I}.$$

For  $\mathbf{m} \in \mathbb{Z}^k$  let  $\mathcal{I}^{(\mathbf{m})}$  be the full subcategory of  $\mathcal{I}$  whose objects are  $\mathcal{T}(\mathfrak{g})$ -modules on which the zero degree central elements  $K_i$ , act by the integer  $m_i$  for  $1 \leq i \leq k$  and let  $\mathcal{I}_{fin}^{(\mathbf{m})}$  be the full subcategory of  $\mathcal{I}^{(\mathbf{m})}$  whose objects are  $\mathcal{T}(\mathfrak{g})$ -modules having finite-dimensional weight spaces.

For  $\beta = \alpha + \delta_{\mathbf{m}} \in R^{re}$  with  $\alpha \in R_{fin}$  let  $r_\beta$  be an operator on an integrable  $\mathcal{T}(\mathfrak{g})$ -module  $V$  defined by  $r_\beta = \exp(x_\beta^+) \exp(-x_\beta^-) \exp(x_\beta^+)$ , with  $x_\beta^\pm = x_\alpha^\pm \otimes t^{\pm \mathbf{m}}$ . Let  $W_{tor} = \langle r_\beta : \beta \in R_{tor}^{re} \rangle$ , be the group generated by the operators  $r_\beta$  for  $\beta \in R_{tor}^{re}$ . By [Kac, Lemma 3.8, §6.5], it is known that  $r_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta$ , for  $\lambda \in \mathfrak{h}_{tor}^*$ . With these notations part (1) of the following is standard and part (2) has been proved in [Kh]. (Refer to [R2, Kh] for details.)

**Lemma.** *Let  $V$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module. Given  $\lambda \in P(V)$ , the following hold.*

- i.  $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ , for  $1 \leq i \leq n+k$ .
- ii.  $w\lambda \in P(V)$  and  $\dim V_\lambda = \dim V_{w\lambda}$  for all  $\lambda \in P(V)$ ,  $w \in W_{tor}$ .
- iii. Let  $\alpha \in R_{fin}^+$  and  $\beta = \alpha + m_i \delta_i \in R_{tor}^{re}$ . Given an integrable  $\mathcal{T}(\mathfrak{g})$ -module  $V$  and  $\lambda \in P(V)$ , we have

$$r_\alpha r_\beta(\lambda) = \lambda + \frac{2}{(\alpha|\alpha)}(m_i \langle \lambda, K_i \rangle) \alpha - (\langle \lambda, \alpha^\vee \rangle + \frac{2}{(\alpha|\alpha)} m_i \langle \lambda, K_i \rangle) \delta_i.$$

In particular, if  $\lambda + \sum_{i=1}^k r_i \delta_i \in P(V)$  is such that  $\langle \lambda, \alpha_{n+1}^\vee \rangle = m$  and  $\langle \lambda, K_j \rangle = 0$  for  $j = 2, \dots, k$ , then there exists  $\mathbf{m} = (m_2, \dots, m_k) \in \mathbb{Z}^{k-1}$  with  $0 \leq m_i < m$  for  $2 \leq i \leq k$  such that  $\lambda + r_1 \delta_1 + \sum_{i=2}^k m_i \delta_i \in P(V)$ .

The following result on integrable  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  was proved in [Kh] and will be used repeatedly in Section 3.]

**Proposition.**

- i. Let  $V$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module. Then  $V = \bigoplus_{\mathbf{m} \in \mathbb{Z}^k} V^{(\mathbf{m})}$  where for each  $\mathbf{m} \in \mathbb{Z}^k$ ,  $V^{(\mathbf{m})} \in \mathcal{I}^{(\mathbf{m})}$ . Up to an isomorphism the direct summand  $V^{(\mathbf{m})}$  of  $V$  is of the form  $V^{(m\mathbf{e}_1)}$  where  $m = \gcd(m_1, \dots, m_k)$ . Further,  $\text{Ext}_{\mathcal{I}}^1(V, U) = 0$  for all  $V \in \text{Ob } \mathcal{I}^{(\mathbf{m})}$  and  $U \in \text{Ob } \mathcal{I}^{(\mathbf{n})}$  whenever  $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^k$  is such that  $\mathbf{m} \neq \mathbf{n}$ . In particular,

$$\mathcal{I} = \bigoplus_{\mathbf{m} \in \mathbb{Z}^k} \mathcal{I}^{(\mathbf{m})},$$

and given  $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$  the category  $\mathcal{I}^{(\mathbf{m})}$  is equivalent to  $\mathcal{I}^{(m\mathbf{e}_1)}$  for  $m = \gcd(m_1, \dots, m_k)$ .

- ii. Let  $V$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$ , where  $m > 0$ . Let  $\mathfrak{n}_{aff}^+$  be the positive root space of the affine Lie subalgebra  $\mathfrak{g}_{aff} = \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}] \oplus \mathbb{C}K_1 \oplus \mathbb{C}d_1$  of  $\mathcal{T}(\mathfrak{g})$ . Then

$$V_{aff}^+ = \{v \in V_\lambda : \mathfrak{n}_{aff}^+ \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v = 0\}$$

is a non-empty subset of  $V$ .

In view of the above result we shall restrict ourselves to the study of the subcategory  $\mathcal{I}^{(me_1)}$  of  $\mathcal{I}$  in the rest of the paper.

### 3. THE CATEGORY $\mathcal{I}_{fin}^{(me_1)}, m > 0$

**3.1.** For  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{C}^k$ , set

$$V\{\mathbf{a}\} = \{v \in V : d_i v = (a_i + k)v \text{ for some } k \in \mathbb{Z}\}.$$

Clearly  $V\{\mathbf{a}\}$  is a  $\mathcal{T}(\mathfrak{g})$ -submodule of  $V$  and  $V\{\mathbf{a}\} = V\{\mathbf{b}\}$  if and only if  $\mathbf{a} - \mathbf{b} \in \mathbb{Z}^k$ . For any  $\bar{\mathbf{a}} \in \mathbb{C}^k / \mathbb{Z}^k$  let  $\mathcal{I}\{\bar{\mathbf{a}}\}$  be the full subcategory of integrable  $\mathcal{T}(\mathfrak{g})$ -modules  $V$  satisfying  $V = V\{\mathbf{a}\}$ , where  $\mathbf{a}$  is any representative of  $\bar{\mathbf{a}}$ . In [CG, Lemma 3.2] the following result was proved for graded level zero integrable representations of affine Lie algebras. The proof for integrable representations of toroidal Lie algebras is analogous.

**Lemma.** Let  $\mathbf{a}, \mathbf{b} \in \mathbb{C}^k$  be such that  $\mathbf{a} - \mathbf{b} \notin \mathbb{Z}^k$ . Then for  $V$  in  $\mathcal{I}\{\bar{\mathbf{a}}\}$  and  $V'$  in  $\mathcal{I}\{\bar{\mathbf{b}}\}$ ,  $\text{Ext}_{\mathcal{I}}^1(V, V') = 0$ . In particular,

$$\mathcal{I} = \bigoplus_{\bar{\mathbf{a}} \in \mathbb{C}^k / \mathbb{Z}^k} \mathcal{I}\{\bar{\mathbf{a}}\}$$

and the categories  $\mathcal{I}\{\bar{\mathbf{a}}\}$  are equivalent for all  $\bar{\mathbf{a}} \in \mathbb{C}^k / \mathbb{Z}^k$ .

Without loss of generality we thus restrict ourselves to the subcategory  $\mathcal{I}\{\bar{\mathbf{0}}\}$  of  $\mathcal{I}$ . That is, in the rest of the paper, by an integrable  $\mathcal{T}(\mathfrak{g})$ -module we will always refer to an object in  $\mathcal{I}\{\bar{\mathbf{0}}\}$ .

**3.2.** For an affine Lie algebra or a  $k$ -toroidal Lie algebra with  $k = 1$ , the following proposition was proved in [CG]. It can be proved in the same manner for a  $k$ -toroidal Lie algebra where  $k \in \mathbb{N}$ .

**Proposition.** Let  $V$  be an object in  $\mathcal{I}_{fin}^{(0)}$ . Then  $V$  is isomorphic to a direct sum of indecomposable modules only finitely many of which are non-trivial.

In this section we prove an analogous result for objects of the category  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ . The following is the main result of this section.

**Theorem.** Let  $V$  be an object in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m \neq 0$ . Then  $V$  is both artinian and noetherian as a  $\mathcal{T}(\mathfrak{g})$ -module. In particular,  $V$  has a composition series and is isomorphic to a direct sum of finitely many indecomposable modules.

Note that the theorem is not true when  $m = 0$  since there exists indecomposable modules of infinite length in  $\mathcal{I}_{fin}$ . For examples refer to [CG, Section 4].

**3.3.** The next proposition plays a vital role in showing that a  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$  is artinian.

**Proposition.** Let  $V$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ . Then  $V$  contains an irreducible  $\mathcal{T}(\mathfrak{g})$ -submodule generated by a weight vector in  $V_{aff}^+$ .

*Proof.* Let  $V$  be an integrable module in  $\mathcal{I}_{fin}^{(me_1)}$ . Then by Proposition 2.5(ii)  $V_{aff}^+$  is non-empty. Let  $v_1 \in V_{aff}^+$  be a non-zero vector of weight  $\lambda_1$  and let  $V_1 = \mathcal{U}(\mathcal{T}(\mathfrak{g}))v_1$  be the  $\mathcal{T}(\mathfrak{g})$ -module generated by  $v_1$ . If  $V_1$  is irreducible, we are done. If not, then there exists a proper non-zero  $\mathcal{T}(\mathfrak{g})$ -submodule

$\mathcal{Y}_1$  of  $V_1$ . Clearly  $v_1 \notin \mathcal{Y}_1$  and  $\mathcal{Y}_1 \in \text{Ob } \mathcal{I}_{fin}^{(me_1)}$ . If  $V$  does not contain any irreducible  $\mathcal{T}(\mathfrak{g})$ -submodule generated by a weight vector in  $V_{aff}^+$ , then repeating the above argument, an infinite chain of  $\mathcal{T}(\mathfrak{g})$ -submodules  $\{V_i\}_{i \geq 1}$  of  $V$  can be obtained such that  $V_i = \mathcal{U}(\mathcal{T}(\mathfrak{g}))v_i$  where  $v_i \in V_{aff}^+$  is a non-zero weight vector of weight  $\lambda_i$  and  $V_{i+1} \subsetneq V_i$  for all  $i \geq 1$ . Further using Lemma 2.5(iii), we may assume that for all  $i$ ,  $0 \leq \lambda_i(d_j) < m$ , for each  $2 \leq j \leq k$  and  $i \geq 1$ .

As the set of  $(k-1)$ -tuples  $(m_2, \dots, m_k) \in \mathbb{Z}^{k-1}$  such that  $0 \leq m_i < m$  is finite, given that the sequence  $\{\lambda_i\}_{i \geq 1}$  is infinite, there must exist  $(m_2, \dots, m_k) \in \mathbb{Z}^{k-1}$  with  $0 \leq m_j < m$  for each  $2 \leq j \leq k$  and an infinite subsequence  $\{\lambda_\ell\}_{\ell \in N_m}$  of  $\{\lambda_i\}_{i \in \mathbb{Z}_+}$  such that for all  $\ell \in N_m$

$$\lambda_\ell(d_j) = m_j, \quad \text{for } 2 \leq j \leq k.$$

Since each weight space of  $V$  is finite-dimensional we may further assume that the subsequence  $\{\lambda_\ell\}_{\ell \in N_m}$  is such that  $\lambda_\ell|_{\mathfrak{h}_{aff}} \succ \lambda_{\ell+1}|_{\mathfrak{h}_{aff}}$ , where  $\lambda_j|_{\mathfrak{h}_{aff}}$  denotes the restriction of  $\lambda_j$  to  $\mathfrak{h}_{aff}$ .

For each  $\ell \in N_m$ , set  $V_{aff}^\ell = \mathcal{U}(\mathfrak{g}_{aff})v_\ell$ . Clearly  $V_{aff}^\ell$  is an integrable  $\mathfrak{g}_{aff}$ -module of positive level and has finite-dimensional weight spaces. Hence by Proposition 2.4(i),  $V_{aff}^\ell$  is isomorphic to a direct sum of irreducible  $\mathfrak{g}_{aff}$ -modules  $X(\lambda_{\ell,s})$  with  $\lambda_{\ell,s} \in P_{aff}^+$ . Choose  $s_\ell$  such that  $\lambda_{\ell,s_\ell} \succ \lambda_{\ell+1}|_{\mathfrak{h}_{aff}}$ . Note such a  $s_\ell$  exists as

$$\lambda_\ell|_{\mathfrak{h}_{aff}} \in P_{aff}(V_{aff}^\ell), \quad \text{and} \quad \lambda_\ell|_{\mathfrak{h}_{aff}} \succ \lambda_{\ell+1}|_{\mathfrak{h}_{aff}}, \quad \forall \ell \in N_m.$$

Further, since  $\lambda_\ell|_{\mathfrak{h}_{aff}} \succ \lambda_{\ell+1}|_{\mathfrak{h}_{aff}}$  for every  $\ell \in N_m$ , there exists a unique  $\gamma \in P_{fin}/Q_{fin}$  such that  $\lambda_\ell|_{\mathfrak{h}_{fin}} \equiv \varpi_\gamma \pmod{Q_{fin}}$  for all  $\ell \in N_m$ . Setting

$$\Lambda_{m,\gamma} = \begin{cases} m\Lambda_{n+1} + \varpi_\gamma & \text{if } \gamma \neq 0 \\ m\Lambda_{n+1} & \text{if } \gamma = 0 \end{cases},$$

it follows from Proposition 2.4(iii) that  $\Lambda_{m,\gamma} + \lambda_{\ell,s_\ell}(d_1)\delta_1 + \sum_{i=2}^k m_i\delta_i \in P_{aff}(\lambda_{\ell,s_\ell})$  for all  $\ell \in N_m$ . In particular if  $r_0^\ell = \min_{\substack{j \in N_m, \\ 1 \leq j \leq \ell}} \{\lambda_{j,s_j}(d_1)\}$ , then by Proposition 2.4(iii),

$$\Lambda_{m,\gamma} + r_0^\ell\delta_1 + \sum_{i=2}^k m_i\delta_i \in P_{aff}(\lambda_{j,s_j}), \quad \forall 1 \leq j \leq \ell, j \in N_m.$$

Since  $\lambda_{j,s_j}$  is distinct for  $j \in N_m$  and the sum of  $X(\lambda_{j,s_j})$  is direct, we see that given any  $\ell \in N_m$  the dimension of the weight space of  $V$  corresponding to the weight  $\Lambda_{m,\gamma} + r_0^\ell\delta_1 + \sum_{i=2}^k m_i\delta_i$  is greater than equal to  $\ell$ . Thus if the sequence  $\{\lambda_i|_{\mathfrak{h}_{aff}}\}_{i \geq 1}$  and hence  $\{\lambda_\ell\}_{\ell \in N_m}$  is infinite, there would exist an infinite-dimensional weight space of  $V$  which would contradict the finite-dimensionality of the weight spaces of  $V$ . This proves the proposition.  $\square$

The following result follows immediately from the above proposition.

**Corollary.** *Let  $V \in \text{Ob } \mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module. Then  $V$  is artinian i.e., there does not exist any infinite properly descending sequence of submodules  $V_1 \supsetneq V_2 \supsetneq \dots$  in  $V$ .*

**3.4.** The next results shows that every object in  $\mathcal{I}_{fin}^{(me_1)}$  is noetherian.

**Proposition.** *Let  $V$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ . Then there does not exist any infinite properly ascending sequence of submodules  $V_1 \subsetneq V_2 \subsetneq \dots$  in  $V$ .*



*Proof.* Every  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$  can be written as the direct sum of its weight spaces. Hence  $V$  can be written as

$$V = \bigoplus_{\gamma \in \Gamma} V^\gamma, \quad \text{where } V^\gamma = \bigoplus_{\mu \equiv \varpi_\gamma \pmod{Q_{fin}}} V_{m\Lambda_{n+1} + \mu + \delta_s}.$$

Since  $\Gamma$  is a finite group and  $V^\gamma$  is a  $\mathcal{T}(\mathfrak{g})$ -module for each  $\gamma \in \Gamma$ , we may assume without loss of generality that  $V = V^\gamma$ . For a contradiction suppose that  $V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots$  is an infinite sequence of properly ascending submodules of  $V$ . Using Proposition 3.3 we may further assume that  $V_i/V_{i-1}$  is a non-trivial simple  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$  for each  $i \in \mathbb{N}$ . Then there exists

$$\lambda_i = m\Lambda_{n+1} + \lambda_i|_{\mathfrak{h}_{fin}} + \sum_{j=1}^k r_j^i \delta_j \in P(V_i/V_{i-1}),$$

with  $m\Lambda_{n+1} + \lambda_i|_{\mathfrak{h}_{fin}} + r_1^i \delta_1 \in P_{aff}^+$  and a non-zero weight vector  $v_i \in (V_i/V_{i-1})_{\lambda_i}$  such that  $V_i/V_{i-1}$  is isomorphic as a  $\mathcal{T}(\mathfrak{g})$ -module to  $\mathcal{U}(\mathcal{T}(\mathfrak{g})).v_i$  for each  $i \in \mathbb{N}$ .

Set  $V_{aff}^i = \mathcal{U}(\mathfrak{g}_{aff}).v_i$ . Clearly  $V_{aff}^i$  is an integrable  $\mathfrak{g}_{aff}$ -module of level  $m$  ( $m > 0$ ) and has finite-dimensional weight spaces. Hence by Proposition 2.4(i) and (iii),  $V_{aff}^i$  can be written as the direct sum of finitely many irreducible  $\mathfrak{g}_{aff}$ -modules  $\{X(\mu_r^i) : 1 \leq r \leq f(i)\}$ , where  $f(i) \in \mathbb{N}$  indicates the number of direct summands of  $V_{aff}^i$ . For each  $i \in \mathbb{N}$ , let  $r_i = \min_{1 \leq r \leq f(i)} \{\mu_r^i(d_1)\}$ . Then by Proposition 2.4(iii)

$$m\Lambda_{n+1} + \varpi_\gamma + s\delta_1 \in P(X(\mu_r^i)), \quad \forall 1 \leq r \leq f(i),$$

whenever  $s \leq r_i$ . Since the sum of the  $\mathfrak{g}_{aff}$ -modules is direct, using Lemma 2.5(iii), it follows that there exists  $(s_2^i, \dots, s_k^i) \in \mathbb{Z}^{k-1}$  with  $0 \leq s_j^i < m$  for  $2 \leq j \leq k$ , such that

$$\dim(V_i/V_{i-1})_{m\Lambda_{n+1} + \varpi_\gamma + s\delta_1 + \sum_{j=2}^k s_j^i \delta_j} \geq f(i) \geq 1, \quad \forall i \in \mathbb{N},$$

whenever  $s \leq r_i$ . As the set  $\mathbb{Z}_m^{k-1} = \{(s_2, \dots, s_k) \in \mathbb{Z}^{k-1} : 0 \leq s_j < m, \forall 2 \leq j \leq k\}$  is finite, there must exist  $(m_2, \dots, m_k) \in \mathbb{Z}_m^{k-1}$  such that  $s_j^i = m_j$  for  $2 \leq j \leq k$  for infinitely many  $i \in \mathbb{N}$ . That is, there exists an infinite subset  $N_m$  of  $\mathbb{N}$  such that for every  $\ell \in N_m$ ,

$$m\Lambda_{n+1} + \varpi_\gamma + s\delta_1 + \sum_{j=2}^k m_j \delta_j \in P(V_\ell/V_{\ell-1}), \quad \forall s \leq r_\ell.$$

This implies that given a subset  $N_l$  of  $N_m$  of cardinality  $l$  if  $r_{N_l} = \min_{i \in N_l} \{r_i\}$ , then the dimension of the weight space of  $V$  corresponding to the weight  $m\Lambda_{n+1} + \varpi_\gamma + r\delta_1 + \sum_{j=2}^k m_j \delta_j$  is greater than equal to  $\ell$  for all  $r \leq r_{N_l}$ . Thus if the sequence  $\{V_i\}_{i \in \mathbb{N}}$  is infinite, there would exist an infinite-dimensional weight space of  $V$  which would contradict the finite-dimensionality of the weight spaces of  $V$ . This completes the proof of the proposition.  $\square$

### 3.5. We now give a proof of Theorem 3.2.

*Proof.* From Corollary 3.3 and Proposition 3.4 we know that every  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  is both artinian and noetherian. Hence every indecomposable object in  $\mathcal{I}_{fin}^{(me_1)}$  has a composition series. To complete the proof of the theorem it suffices to show that every  $\mathcal{T}(\mathfrak{g})$ -module  $V$  in  $\mathcal{I}_{fin}^{(me_1)}$  is the

direct sum of finitely many indecomposable  $\mathcal{T}(\mathfrak{g})$ -modules. As in Proposition 3.4 we can write

$$V = \bigoplus_{\gamma \in \Gamma} V^\gamma, \quad \text{where } V^\gamma = \bigoplus_{\mu \equiv \varpi_\gamma \pmod{Q}} V_{m\Lambda_{n+1} + \mu + \delta_s}.$$

Since  $\Gamma$  is finite, it suffices to prove that  $V^\gamma$  is the direct sum of finitely many indecomposable  $\mathcal{T}(\mathfrak{g})$ -modules for each  $\gamma \in \Gamma$ . For a contradiction suppose that  $V^\gamma$  is not the direct sum of finitely many indecomposable modules. i.e., in particular,  $V^\gamma$  is not indecomposable. Hence it can be written as the direct sum of two  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}^{(me_1)}$ , namely  $V^\gamma = M_1 \oplus M_2$ . Repeating the argument we obtain an strictly descending chain  $\cdots \subseteq N_2 \subseteq N_1 = M_i \subseteq N_0 = V^\gamma$  of  $\mathcal{T}(\mathfrak{g})$ -submodules of  $V^\gamma$  such that  $N_i \in \text{Ob } \mathcal{I}_{fin}^{(me_1)}$  and  $N_i$  is not isomorphic to a direct sum of finitely many indecomposable  $\mathcal{T}(\mathfrak{g})$ -modules for each  $i \in \mathbb{N}$ . By Corollary 3.3 such a chain terminates, i.e., there exists  $r \in \mathbb{N}$  such that  $N_r = N_p$  for all  $p > r$ . This contradicts our assumption on  $N_r$ . Thus it follows that  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  is the direct sum of finitely many indecomposables.  $\square$

**Remark.** Given an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ , by [R1, Kh] the set of non-zero graded homogeneous central elements  $\{K_j t^{\mathbf{r}} : 2 \leq j \leq k, \mathbf{r} \in \mathbb{Z}^k\}$  act trivially on  $V$  and  $K_1 t^{\mathbf{r}}$  acts trivially on  $V$  whenever  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{Z}^k$  is such that  $r_1 \neq 0$ . Thus any irreducible module for  $\mathcal{T}(\mathfrak{g})$  is in fact an irreducible representation of the Lie algebra  $\bar{\mathcal{T}}(\mathfrak{g}) := \mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus D_k$  where  $\tilde{\mathcal{Z}}$  is the subspace of  $\mathcal{Z}$  spanned by the central elements  $\{K_1 t^{\mathbf{r}} : \mathbf{r} = (r_2, \dots, r_k) \in \mathbb{Z}^{k-1}\}$ . By Theorem 3.2 every indecomposable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$  has a composition series. Hence applying induction on the length of an indecomposable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$  it can be seen that every such module is in fact a representation of the Lie algebra  $\bar{\mathcal{T}}(\mathfrak{g})$ .

**3.6.** Given an abelian category  $\mathcal{C}$  in which every object can be decomposed as direct sum of indecomposable objects, we say two indecomposable objects  $U_1$  and  $U_2$  of  $\mathcal{C}$  are linked and write  $U_1 \sim U_2$  if there exists a family of indecomposable objects  $U_1 = U, U_2, \dots, U_r = V$  in  $\mathcal{C}$  such that either  $\text{hom}_{\mathcal{C}}(U_i, U_{i+1}) \neq 0$  or  $\text{hom}_{\mathcal{C}}(U_{i+1}, U_i) \neq 0$  for all  $i = 1, \dots, r-1$ . It is easy to see that  $\sim$  defines an equivalence relation on  $\mathcal{C}$ . The corresponding equivalence classes are called the blocks in  $\mathcal{C}$ . Each block is a full abelian subcategory and the category  $\mathcal{C}$  is a direct sum of the blocks.

Let  $\mathcal{C} = \mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ . By Theorem 3.2 it is clear that every object in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  has a Jordan Holder series and hence the following results are standard in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ . We shall repeatedly use these results from [CM] without further mention.

**Proposition.**

- i. Any sequence  $0 \subset V_1 \subset \cdots \subset V_k \subset V$  of  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m \neq 0$  can be refined to a Jordan-Holder series of  $V$ .
- ii. Suppose that  $0 \subset U_1 \subset \cdots \subset U_r = U$  and  $0 \subset V_1 \subset \cdots \subset V_s \subset V$  are Jordan-Holder series for  $U$  and  $V$  in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m \neq 0$  and  $\mathcal{I}_{fin}^{(se_1)}$ ,  $s \neq 0$  respectively. Then the irreducible constituents of  $U \otimes V$  occur as constituents of  $U_k \otimes V_\ell$  for some  $1 \leq k \leq r$  and  $1 \leq \ell \leq s$ .
- iii. Let  $U_1, U_2, U_3$  be  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}$  such that  $U_1$  is linked to  $U_2$ . Then  $U_1 \otimes U_3$  is linked to  $U_2 \otimes U_3$ .

In [CM] a parametrization for the blocks in the category of finite-dimensional representations of the affine Lie algebras has been given. Using their results a parametrization has been obtained in [CG] for the blocks in the category of integrable level zero representations with finite-dimensional weight spaces

of the affine Kac-Moody Lie algebras. Following [CM] and [CG] we give in Section 5 a parametrization of the blocks in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ .

#### 4. IRREDUCIBLE OBJECTS IN $\mathcal{I}_{fin}^{(me_1)}$ , $m > 0$

We have seen that every indecomposable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  has a composition series. Hence the irreducible objects of the category  $\mathcal{I}_{fin}^{(me_1)}$  play a vital role in determining its blocks. In this section we recall the results from [R2, R3, Kh] that give a parametrization of the irreducibles in  $\mathcal{I}_{fin}^{(me_1)}$ .

**4.1.** Let  $\Pi$  be the monoid of finitely supported functions  $\pi : \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow P_{aff}^+$ , where given  $\pi, \pi' \in \Pi$  and  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  we define

$$(\pi + \pi')(M) = \pi(M) + \pi'(M), \quad \text{supp}(\pi) = \{M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] : \pi(M) \neq 0\},$$

and

$$\text{wt}(\pi) = \sum_{M \in \text{supp}(\pi)} \pi(M).$$

Given  $\pi \in \Pi$ , let  $M_1, M_2, \dots, M_l$  be an enumeration of  $\text{supp}(\pi)$  and let

$$X_\pi = \otimes_{i=1}^l X(\pi(M_i)),$$

be the  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module on which the action of the Lie algebra is defined by :

$$Y \otimes f.v_1 \otimes \dots \otimes v_l = \sum_{i=1}^l \text{ev}_{M_i}(f)v_1 \otimes \dots \otimes Y.v_i \otimes \dots \otimes v_l,$$

where  $Y \in \mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}] \oplus \mathbb{C}K_1 \oplus \mathbb{C}d_1$ ,  $f \in \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ ,  $v_i \in X(\pi(M_i))$  and  $\text{ev}_{M_i} : \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbb{C}$  is the evaluation map at the point in  $(\mathbb{C}^*)^{k-1}$  corresponding to the maximal ideal  $M_i$  of  $\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  for  $1 \leq i \leq l$ . Let

$$L(X_\pi) = X_\pi \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}],$$

be the  $\mathcal{T}(\mathfrak{g})$ -module on which the Lie algebra action is defined by :

$$Y \otimes f.(w \otimes f') = (Y \otimes f.w) \otimes f f', \quad K_j t^{\mathbf{m}}.(w \otimes f') = 0, \quad \forall 2 \leq j \leq k, \mathbf{m} \in \mathbb{Z}^k,$$

$$d_i.(w \otimes f') = w \otimes d_i(f'), \quad \text{for } 2 \leq i \leq k, \quad d_1.w \otimes f' = d_1(w) \otimes f',$$

for  $Y \in \mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}] \oplus \mathbb{C}K_1$ ,  $f, f' \in \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  and  $w \in X_\pi$ . For  $M \in \text{supp}(\pi)$ , let  $v_M$  be the highest weight vector of  $X(\pi(M))$  and let

$$v_\pi := v_{M_1} \otimes v_{M_2} \otimes \dots \otimes v_{M_l}.$$

Clearly  $v_\pi \in L(X_\pi)_{aff}^+$  and  $\mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]).v_\pi$  is a  $\mathbb{Z}^{k-1}$ -graded subset of  $L(X_\pi)_{aff}^+$ . Setting

$$\mathbf{A}_\pi := \mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]).v_\pi$$

it is easy to see that  $\mathbf{A}_\pi$  is a  $\mathbb{Z}^{k-1}$ -graded algebra, that is,

$$\mathbf{A}_\pi = \bigoplus_{\mathbf{m} \in \mathbb{Z}^{k-1}} \mathbf{A}_\pi[\mathbf{m}],$$

where  $\mathbf{A}_\pi[\mathbf{m}] = \{h_1 \otimes t^{\mathbf{m}_1} h_2 \otimes t^{\mathbf{m}_2} \dots \in \mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]) \text{ with } \sum_i \mathbf{m}_i = \mathbf{m}\}$  for each  $\mathbf{m} \in \mathbb{Z}^{k-1}$ . For  $\pi \in \Pi$  let

$$G_\pi := \{\mathbf{m} = (m_2, \dots, m_k) \in \mathbb{Z}^{k-1} : A_\pi[\mathbf{m}] \neq 0\}.$$

From the definition of the action of  $\mathcal{T}(\mathfrak{g})$  on  $L(X_\pi)$  it is clear that

$$h \otimes t^{\mathbf{m}}.v_\pi = \left( \sum_{i=1}^r \pi(M_i)(h) \text{ev}_{M_i}(t^{\mathbf{m}}) \right) v_\pi \otimes t^{\mathbf{m}} = 0, \quad \forall \mathbf{m} \in \mathbb{Z}^{k-1} - G_\pi, \quad (4.1)$$

that is,

$$\left( \sum_{i=1}^r \pi(M_i)(h) \text{ev}_{M_i}(t^{\mathbf{m}}) \right) = 0, \quad \forall \mathbf{m} \in \mathbb{Z}^{k-1} - G_\pi.$$

It has been shown in [Kh] that the set  $G_\pi$  is a subgroup of  $\mathbb{Z}^{k-1}$  of finite index and we refer to  $G_\pi$  as the group associated with  $\pi \in \Pi$ . Let  $G^\pi = \mathbb{Z}^{k-1}/G_\pi$  be the corresponding quotient group.

The following results have been proved in [R2, Proposition 3.5, Theorem 3.18, Example 4.2], [R3].

**Proposition.** *For  $\pi \in \Pi$ , let  $v_\pi$  be the highest weight vector of  $X_\pi$ , let  $G_\pi$  be the group associated to  $\pi$  and  $G^\pi = \mathbb{Z}^{k-1}/G_\pi$ . Then we have the following.*

i. *For each  $\mathbf{g} \in G^\pi$ , the  $\mathcal{T}(\mathfrak{g})$ -module*

$$X_\pi^{\mathbf{g}} = \mathcal{U}(\mathcal{T}(\mathfrak{g})).v_\pi \otimes t^{\mathbf{g}},$$

*is an irreducible  $\mathcal{T}(\mathfrak{g})$ -module.*

ii.  *$L(X_\pi)$  is completely reducible as a  $\mathcal{T}(\mathfrak{g})$ -module. In fact as a  $\mathcal{T}(\mathfrak{g})$ -module  $L(X_\pi)$  is isomorphic to the direct sum of the irreducible  $\mathcal{T}(\mathfrak{g})$ -modules  $X_\pi^{\mathbf{g}}, \mathbf{g} \in G^\pi$  that is,*

$$L(X_\pi) \cong_{\mathcal{T}(\mathfrak{g})} \bigoplus_{\mathbf{g} \in G^\pi} X_\pi^{\mathbf{g}}.$$

Further if  $V$  is an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}, m > 0$ , then there exists  $\pi \in \Pi$  with  $\text{wt}(\pi) = \Lambda \in P_{aff}^+$  satisfying  $\Lambda(\alpha_{n+1}) = m, m > 0$  and  $\mathbf{g} \in G^\pi$  such that  $V$  is isomorphic to  $X_\pi^{\mathbf{g}}$ .

**Remark.** From the description of the action of  $\mathcal{T}(\mathfrak{g})$  on  $L(X_\pi)$  it is clear that for all  $\pi \in \Pi$ , the vector space  $X_\pi$  is a representation of the Lie algebra  $\bigoplus_{M \in \text{supp}(\pi)} (\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M \oplus \tilde{\mathcal{Z}}) \oplus \mathbb{C}d_1$ .

This fact shall be used in Section 5.

**4.2.** Notice that for  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbb{C}^*)^{k-1}$  the map  $s_{\mathbf{b}} : \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  given by  $t_i \mapsto b_i t_i, i = 2, \dots, k$  is an isomorphism. Denote by  $\mathbf{b}.M$  the image of a maximal ideal  $M$  of  $\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  under the isomorphism  $s_{\mathbf{b}}$ . Define an action of  $(\mathbb{C}^*)^{k-1}$  on  $\Pi$  by:

$$\mathbf{b}.\pi(M) = \pi(\mathbf{b}.M), \quad \text{for all } M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].$$

With this notation we have the following result from [Kh] on the isomorphism classes of irreducible representations of  $\mathcal{T}(\mathfrak{g})$ .

**Proposition.** *Given  $\pi, \pi' \in \Pi$ ,  $\mathbf{g} \in G^\pi$  and  $\mathbf{g}' \in G^{\pi'}$ , the irreducible  $\mathcal{T}(\mathfrak{g})$ -modules  $X_\pi^{\mathbf{g}}$  and  $X_{\pi'}^{\mathbf{g}'}$  are isomorphic if and only if there exists  $\mathbf{b} \in (\mathbb{C}^*)^{k-1}$  such that*

- i.  $\text{supp}(\pi') = \{\mathbf{b}.M : M \in \text{supp}(\pi)\}$ .
- ii. *For each  $M \in \text{supp}(\pi)$ , there exists one-dimensional  $\mathfrak{g}_{aff}$ -module  $Z_M$  such that  $X(\pi(M)) \otimes Z_M$  is isomorphic to  $X(\pi'(\mathbf{b}.M))$  as a  $\mathfrak{g}_{aff}$ -module.*
- iii.  $\mathbf{g} \equiv \mathbf{g}' \pmod{G_\pi}$ .

Since for  $\pi \in \Pi$ ,  $X_\pi$  is a  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module the following is an immediate consequence of Proposition 4.2.

**Corollary.** *Given  $\pi, \pi' \in \Pi$ , the irreducible  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules  $X_\pi$  and  $X_{\pi'}$  are isomorphic if and only if  $\pi' = \mathbf{b}.\pi$  for some  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbb{C}^*)^{k-1}$ . i.e., there exists  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbb{C}^*)^{k-1}$  such that  $\text{supp}(\pi_2) = \{\mathbf{b}.M : M \in \text{supp}(\pi_1)\}$ , and for each  $M \in \text{supp}(\pi_1)$ , there exists one-dimensional  $\mathfrak{g}_{aff}$ -module  $Z_M$  such that  $X(\pi_1(M)) \otimes Z_M$  is isomorphic to  $X(\pi_2(\mathbf{b}.M))$  as a  $\mathfrak{g}_{aff}$ -module.*

## 5. SPECTRAL CHARACTERS OF $\mathcal{I}_{fin}^*$

In this section we continue with our study of the structure of the category  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$ ,  $m > 0$  and give a description of its blocks.

**5.1.** Let  $\Xi$  be the set of all finitely supported functions  $\xi : \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbb{Z} \times \Gamma$ . Since  $\mathbb{Z}$  and  $\Gamma$  are abelian groups with respect to addition operation, regarding  $\mathbb{Z} \times \Gamma$  as the direct product of abelian groups, it is easy to see that addition of functions defines on  $\Xi$  the structure of an additive abelian group in an obvious way. Denoting the images of the fundamental weights  $\{\omega_i\}_{1 \leq i \leq n}$  in  $\Gamma$  by  $\{\bar{\omega}_i\}_{1 \leq i \leq n}$ , define

$$\begin{aligned} \xi_{i,M}(S) &= \begin{cases} (0, \bar{\omega}_i) & \text{if } S = M, \\ (0, 0) & \text{otherwise} \end{cases} \quad \forall 1 \leq i \leq n, \\ \xi_{n+1,M}(S) &= \begin{cases} (1, 0) & \text{if } S = M, \\ (0, 0) & \text{otherwise} \end{cases} \end{aligned}$$

for  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ . Clearly  $\Xi$  is a free abelian group generated by the set of elements  $\{\xi_{i,M} : 1 \leq i \leq n+1, M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]\}$ . Define an action of  $(\mathbb{C}^*)^{k-1}$  on  $\Xi$  by

$$(\mathbf{b}.\xi)(S) = \xi(\mathbf{b}.S),$$

where  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbb{C}^*)^{k-1}$  and  $S \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ . Let  $\bar{\Xi}$  be the set of orbits in  $\Xi$  under this action. Let  $\Xi_0$  be the subgroup of  $\Xi$  generated by the set of functions

$$\{\xi_{i,M} : 1 \leq i \leq n, M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]\},$$

and let  $\bar{\Xi}_0$  be the set of orbits in  $\Xi_0$  under the given action of  $(\mathbb{C}^*)^{k-1}$  on  $\Xi_0$ .

The following proposition was proved in [CG] in the case when  $k = 1$ . For an arbitrary  $k$  can be obtained in the same way using results from [NS].

**Proposition.** *The full subcategory  $\mathcal{I}_{fin}^{(0)}$  of  $\mathcal{I}_{fin}$  on which the central elements act trivially has a block decomposition. The blocks in  $\mathcal{I}_{fin}^{(0)}$  are parametrized by the elements of  $\bar{\Xi}_0$ . In particular if  $V$  is an indecomposable module in  $\mathcal{I}_{fin}^{(0)}$  such that  $\lambda \in P(V)$  for some  $\lambda \in Q_{fin}$  then  $V$  is linked to the one-dimensional  $\mathcal{T}(\mathfrak{g})$ -module  $\mathbb{C}_{\delta_{\mathbf{m}}}$ ,  $\mathbf{m} \in \mathbb{Z}^k$ , with highest weight  $\sum_{i=1}^k m_i \delta_i$ .*

**5.2.** Given  $\Lambda \in P_{aff}^+$  and  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ , let  $\pi_{\Lambda, M} \in \Pi$  be the function such that

$$\text{supp}(\pi_{\Lambda, M}) = \{M\} \quad \text{and} \quad \pi_{\Lambda, M}(M) = \Lambda.$$

Let  $X(\Lambda, M)$  denote the  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module  $X_{\pi_{\Lambda, M}}$  and let  $v_{\Lambda, M}$  denote its highest weight vector. For a fixed  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  define  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module action on the  $\mathfrak{g}_{aff}$ -module  $X^*(\Lambda)$  by :

$$Y \otimes f.v = ev_M(f)Y.v, \quad \text{where } v \in X^*(\Lambda), Y \in \mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}] \oplus \mathbb{C}K_1 \oplus \mathbb{C}d_1, f \in \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].$$

We denote this  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module by  $X^*(\Lambda, M)$ . Finally for  $r \in \mathbb{Z}$  let  $C_{r\delta_1, M}$  be the one-dimensional  $\mathfrak{g}_{aff}$ -module  $\mathbb{C}_{r\delta_1}$  when regarded as a  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$  at the maximal ideal  $M$ .

We use the parametrization of the blocks in  $\mathcal{I}_{fin}^{(0)}$  to describe the blocks in the category  $\mathcal{I}_{fin}^{(me_1)}$  for  $m \neq 0$ . The following lemma is crucial in this regard.

**Lemma.** *Let  $\Lambda, \mu \in P_{aff}^+$  be such that  $\Lambda(\alpha_{n+1}) = \mu(\alpha_{n+1}) = m > 0$  and  $(\Lambda - \mu)|_{\mathfrak{h}_{fin}} \in Q_{fin}$ . Then for any  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ ,  $X(\Lambda, M)$  is linked to  $X(\mu, M)$  in  $\mathcal{I}_{fin}^{(me_1)}$ .*

*Proof.* First of all note that given  $\Lambda \in P_{aff}^+$ , by Proposition 2.4(v) the module  $X(\Lambda) \otimes X^*(\Lambda)$  is isomorphic to a one-dimensional representation of  $\mathfrak{g}_{aff}$ . Hence regarding the  $\mathfrak{g}_{aff}$ -modules as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules via evaluation map at a maximal ideal  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  we see that  $X(\Lambda, M) \otimes X^*(\Lambda, M)$  is linked to the one-dimensional  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module,  $\mathbb{C}_{r\delta_1}$ ,  $r \in \mathbb{Z}$  in  $\mathcal{I}_{fin}$ . In particular,

$$X(\Lambda, M) \otimes X^*(\mu, M) \otimes X(\mu, M) \sim X(\Lambda, M) \otimes \mathbb{C}_{s\delta_1, M}.$$

On the other hand given  $\Lambda, \mu \in P_{aff}^+$  with  $\Lambda(\alpha_{n+1}^\vee) = m = \mu(\alpha_{n+1}^\vee)$ , by Proposition 2.4(v),  $X(\Lambda) \otimes X^*(\mu)$  is linked to  $V(w((\Lambda - \mu)|_{\mathfrak{h}_{fin}}) \otimes \mathbb{C}[t_1^{\pm 1}])$  for some  $w \in W_{fin}$  such that  $w((\Lambda - \mu)|_{\mathfrak{h}_{fin}}) \in P_{fin}^+$ . If  $(\Lambda - \mu)|_{\mathfrak{h}_{fin}} \in Q_{fin}$  then applying Proposition 5.1 in the case  $k = 1$  we see that  $V(w((\Lambda - \mu)|_{\mathfrak{h}_{fin}}) \otimes \mathbb{C}[t_1^{\pm 1}])$  is linked to  $\mathbb{C}_{n\delta_1}$  for  $n \in \mathbb{Z}$ . Again, considering the  $\mathfrak{g}_{aff}$ -modules as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules via evaluation map at  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  we see that

$$(X(\Lambda, M) \otimes X^*(\mu, M)) \otimes X(\mu, M) \sim \mathbb{C}_{s\delta_1, M} \otimes X(\mu, M).$$

As  $\mathbb{C}_{n\delta_1, M} \otimes X(\mu, M)$  is isomorphic to  $X(\mu, M)$  for all  $n \in \mathbb{Z}$ ,  $\mu \in P_{aff}^+$ , using transitivity of the linking relation, we conclude that the irreducible  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules  $X(\Lambda, M)$  and  $X(\mu, M)$  are linked in  $\mathcal{I}_{fin}^{(me_1)}$ , whenever  $(\Lambda - \mu)|_{\mathfrak{h}_{fin}} \in Q_{fin}$ .  $\square$

Note that the property of the affine Kac-Moody algebras which plays an important role in the above result and hence in the parametrization of the blocks in  $\mathcal{I}_{fin}^{(me_1)}$  is the fact that  $\mathfrak{g}_{aff}/[\mathfrak{g}_{aff}, \mathfrak{g}_{aff}] \neq 0$  and therefore there exists non-trivial one-dimensional representations of the Lie algebra  $\mathfrak{g}_{aff}$ .

**5.3.** Define a map  $\chi : \Pi \rightarrow \Xi$  as follows. For  $\pi \in \Pi$ , let

$$\chi(\pi)(M) = \begin{cases} (\pi(M)(\alpha_{n+1}^\vee), \pi(M)|_{\mathfrak{h}_{fin}} \mod Q_{fin}), & \text{if } M \in \text{supp}(\pi) \\ (0, 0), & \text{otherwise} \end{cases}$$

Given  $\xi \in \Xi$ , let  $\mathcal{I}_{fin}^\xi$  be the full subcategory of  $\mathcal{I}_{fin}$  consisting of integrable  $\mathcal{T}(\mathfrak{g})$ -modules  $V$  such that there exists an irreducible subrepresentation  $X_\pi^\mathfrak{g}$  of  $V$  with  $\pi \in \Pi$ ,  $\mathfrak{g} \in G_\pi$  satisfying the condition  $\chi(\pi) = \xi$ .

**Proposition.** *Let  $\pi_1, \pi_2 \in \Pi$ . Then the  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules  $X_{\pi_1}$ ,  $X_{\pi_2}$  are linked in  $\mathcal{I}_{fin}$  if and only if  $\chi(\pi_1) = \chi(\pi_2)$ .*

*Proof.* Given  $\pi_1, \pi_2 \in \Pi$ , we prove by applying induction on the cardinality  $\#\text{supp}(\pi_1)$  of  $\text{supp}(\pi_1)$  that two  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules  $X_{\pi_1}$  and  $X_{\pi_2}$  are linked in  $\mathcal{I}_{fin}$  whenever  $\chi(\pi_1) = \chi(\pi_2)$ .

Suppose  $\#\text{supp}(\pi_1) = 1$ . Then upto a scaling  $\text{supp}(\pi_1) = \text{supp}(\pi_2) = \{M\}$ , for some  $M \in \max \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ . Since  $\pi_1(M)(\alpha_{n+1}^\vee) = \pi_2(M)(\alpha_{n+1}^\vee)$  and  $(\pi_1(M) - \pi_2(M))|_{\mathfrak{h}_{fin}} \in Q_{fin}$ , it follows from Lemma 5.2 that  $X_{\pi_1} \sim X_{\pi_2}$  as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules.

Assume that the result holds whenever  $\#\text{supp}(\pi_1) \leq k-1$ . As  $\chi(\pi_1) = \chi(\pi_2)$  upto a scaling  $\text{supp}(\pi_1) = \text{supp}(\pi_2)$  and for every  $M \in \text{supp}(\pi_1)$ ,

$$\pi_1(M)(\alpha_{n+1}^\vee) = \pi_2(M)(\alpha_{n+1}^\vee), \quad (\pi_1(M) - \pi_2(M))|_{\mathfrak{h}_{fin}} \in Q_{fin}.$$

Let  $S \in \text{supp}(\pi_i)$  be such that  $\pi_i(S) = \Lambda_i$  and let  $\pi'_i \in \Pi$  be such that  $\pi_i = \pi'_i + \pi_{\Lambda_i, S}$ , for  $i = 1, 2$ . By the first step of induction,  $X(\Lambda_1, S) \sim X(\Lambda_2, S)$  and by inductive hypothesis  $X_{\pi'_1} \sim X_{\pi'_2}$  in  $\mathcal{I}_{fin}$ . Using transitivity of the linking relation it thus follows that  $X_{\pi'_1} \otimes X(\Lambda_1, S) \sim X_{\pi'_2} \otimes X(\Lambda_2, S)$  in  $\mathcal{I}_{fin}$ . i.e.,  $X_{\pi_1} \sim X_{\pi_2}$  whenever  $\chi(\pi_1) = \chi(\pi_2)$ .

Conversely suppose that  $\pi_1, \pi_2 \in \Pi$  are such that  $\chi(\pi_1) \neq \chi(\pi_2)$ . Then either of these conditions hold.

- (i). There exists  $M \in \text{supp}(\pi_1)$  such that  $M \notin \text{supp}(\pi_2)$ .
- (ii). There exists  $M \in \text{supp}(\pi_1) \cap \text{supp}(\pi_2)$  such that either  $\pi_1(M)(\alpha_{n+1}^\vee) \neq \pi_2(M)(\alpha_{n+1}^\vee)$  or  $(\pi_1(M) - \pi_2(M))|_{\mathfrak{h}_{fin}} \notin Q_{fin}$ .

By definition  $\chi(\pi)(M) = (0, 0)$  whenever  $M \notin \text{supp}(\pi)$ . Hence it suffices to prove the proposition when  $\text{supp}(\pi_1) = \text{supp}(\pi_2)$  and we are in case (ii).

Let  $M_1, \dots, M_r$  be an enumeration of  $\text{supp}(\pi_i)$  for  $i = 1, 2$ . For a contradiction assume there exists an indecomposable  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module  $V$  such that

$$0 \rightarrow X_{\pi_2} \rightarrow V \rightarrow X_{\pi_1} \rightarrow 0 \quad (5.1)$$

is a non-split short exact sequence in  $\mathcal{I}_{fin}$ . Since  $X_{\pi_1}$  and  $X_{\pi_2}$  are irreducible modules for the Lie algebra  $\bigoplus_{i=1}^r (\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M_i \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ , for every ideal  $I$  of  $\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  such that  $\cap_{i=1}^r M_i \subseteq I$ , (5.1) is a non-split short exact sequence of  $(\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes I/(\cap_{i=1}^r M_i) \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ . In particular, (5.1) is a non-split short exact sequence for

$$\mathcal{T}^j := (\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes I_j/(\cap_{i=1}^r M_i) \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1, \quad \text{where } I_j = \bigcap_{\substack{i=1 \\ i \neq j}}^r M_i, \quad \text{for } j = 1, 2, \dots, r.$$

This implies that the  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module  $V$  is of the form  $V = V_1 \otimes V_2 \otimes \dots \otimes V_r$ , where  $V_j$  is an indecomposable  $\mathcal{T}^j$ -module for each  $i \in \{1, \dots, r\}$ . Suppose

$$\pi_i(M_j) = \lambda_j^i, \quad \text{for } i = 1, 2, \quad 1 \leq j \leq r,$$

then using the fact that tensor products preserve direct sums it follows that

$$0 \rightarrow X(\lambda_j^2, M_j) \xrightarrow{\iota_j} V_j \xrightarrow{p_j} X(\lambda_j^1, M_j) \rightarrow 0, \quad (5.2)$$

is a non-split short exact sequence of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M_j \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules for  $j = 1, \dots, r$ . Thus there exists  $v_j \in V_j$  such that  $p_j(v_j) = v_{\lambda_j^1, M}$ . Consequently,

$$n_{aff}^+ \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v_j = 0, \quad h.v_j = \lambda_j^1(h)v_j, \quad \forall h \in \mathfrak{h}_{aff}, \quad K_j t^{\mathbf{r}}.v_j = 0, \quad \forall j > 2, \quad \mathbf{r} \in \mathbb{Z}^k,$$

and  $V_j \in \mathcal{I}_{fin}^{(n_j \mathbf{e}_1)}$  for  $1 \leq j \leq r$  where  $n_j \in \mathbb{Z} - \{0\}$  is such that  $\lambda_j^1(\alpha_{n+1}^\vee) = n_j = \lambda_j^2(\alpha_{n+1}^\vee)$ . But  $\chi(\pi_1) \neq \chi(\pi_2)$ , so  $(\lambda_j^1 - \lambda_j^2)|_{\mathfrak{h}_{fin}} \notin Q_{fin}$  for some  $j \in \{1, 2, \dots, r\}$ . Without loss of generality we may assume that  $(\pi_1(M_1) - \pi_2(M_1))|_{\mathfrak{h}_{fin}} \notin Q_{fin}$ . Since  $\iota_1$  and  $p_1$  are  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -module homomorphisms and

$$\text{wt}(X(\lambda_1^2)) \subseteq \lambda_1^2 - Q_{aff}^+, \quad \text{wt}(\mathcal{U}((\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M_1).v_1) \subseteq \lambda_1^1 - Q_{aff}^+,$$

by the exactness of (5.2) it follows that  $\ker p \cap \mathcal{U}((\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M_1).v_1 = 0$ . This implies that  $\mathcal{U}((\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M_1).v_1$  is a direct summand of  $V_1$  which is isomorphic to  $X(\lambda_1^1, M_1)$  thereby contradicting the assumption that  $V_1$  is indecomposable. Hence it follows that  $\text{Ext}_{\mathcal{I}_{fin}}^1(X_{\pi_1}, X_{\pi_2}) = 0$  whenever  $\chi(\pi_1) \neq \chi(\pi_2)$ .  $\square$

Applying induction on the length of a module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m \neq 0$ , it is easy to see that part (i) of the following result is a consequence of the above proposition. Using Proposition 4.1 and the fact that tensor products preserve short exact sequences, it is easy to prove part(ii) as a simple corollary of Proposition 5.3.

**Corollary.**

- i. Given  $\xi \in \Xi$ , let  $\mathcal{I}_{fin}^{\bar{\xi}}$  be a subcategory of  $\mathcal{I}_{fin}^{(me_1)}$  for some  $m > 0$ . Then for all  $\mathcal{T}(\mathfrak{g})$ -modules  $V \in \mathcal{I}_{fin}^{\bar{\xi}}$ ,

$$\text{Ext}_{\mathcal{I}_{fin}}^1(V, X_{\pi}^{\mathbf{g}}) = 0, \quad \forall \mathbf{g} \in G_{\pi},$$

whenever  $\chi(\pi) \neq \xi$ .

- ii. Let  $\pi_1, \pi_2 \in \Pi$  be such that  $\chi(\pi_1) \neq \chi(\pi_2)$ . Then

$$\text{Ext}_{\mathcal{I}_{fin}}^1(X_{\pi_1}^{\mathbf{g}_1}, X_{\pi_2}^{\mathbf{g}_2}) = 0, \quad \forall \mathbf{g}_i \in G^{\pi_i}, \quad i = 1, 2.$$

**5.4.** We are now in a position to give a parametrization of the blocks in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ .

**Definition.** Given  $\pi \in \Pi$ , we shall say

- i.  $\pi$  is of type **I** if there exists  $\pi' \in \Pi$  such that  $\chi(\pi) = \chi(\pi')$  but  $X_{\pi}$  and  $X_{\pi'}$  are not isomorphic as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules.
- ii.  $\pi$  is of type **II** if for all  $\pi' \in \Pi$  satisfying  $\chi(\pi) = \chi(\pi')$ ,  $X_{\pi}$  and  $X_{\pi'}$  are isomorphic as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules.

In order to give examples of finitely supported functions of type **I** and type **II** we recall the Dynkin diagrams of the non-twisted affine Kac-Moody Lie algebras from [Kac] in Table 1. With reference to these labeled Dynkin diagrams consider the following two examples.

Example 1. Suppose  $\mathfrak{g}_{aff}$  is of type  $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}$  or  $E_7^{(1)}$  and  $\pi \in \Pi$  is such that  $\pi(M) = \Lambda_i$ , with  $i \in J_0$  for each  $M \in \text{supp}(\pi)$ . Then  $\chi(\pi)(M) = (1, \bar{\omega}_i)$  for  $M \in \text{supp}(\pi)$ . If  $\pi' \in \Pi$  is such that  $\chi(\pi') = \chi(\pi)$  then  $\pi'(M) = (1, \bar{\omega}_i)$  for each  $M \in \text{supp} \pi'$ . Since for  $i \in J_0$ ,  $\Lambda_i \in P_{aff}^+$  is the only dominant integral weight in  $P_{aff}^+$  such that  $\Lambda_i(\alpha_{n+1}^\vee) = 1$  and  $\Lambda_i|_{\mathfrak{h}_{fin}} \equiv \omega_i \pmod{Q_{fin}}$  by Corollary 4.2  $X_{\pi}$  and  $X_{\pi'}$  are isomorphic as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules. This gives an example of a function in  $\Pi$  which is of type **II**. A study of the diagrams above shows that such cases arise when the associated finite-dimensional Lie algebra  $\mathfrak{g}_{fin}$  is simply-laced and the corresponding finite group  $\Gamma$  is non-trivial.

Example 2. As above let  $\mathfrak{g}_{aff}$  is of type  $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}$  or  $E_7^{(1)}$ . It is known that the highest root  $\theta$  for the corresponding Lie algebras of finite type are given by  $\omega_1 + \omega_n$  for  $\mathfrak{g}_{fin}$  of type  $A_n$ ,  $\omega_2$  for  $\mathfrak{g}_{fin}$  is of type  $D_n, n \geq 4$  and  $E_6$  and  $\omega_1$  for  $\mathfrak{g}_{fin}$  is of type  $E_7$ . Hence for affine Lie algebras of the types specified, it is clear from the given labeled Dynkin diagrams that  $m\Lambda_{n+1} + \theta + \delta_{\mathbf{m}} \in P(V)$  for  $V \in \text{Ob} \mathcal{I}_{fin}^{(me_1)}$  only if  $m \geq 2$ . Thus if  $\pi \in \Pi$  is such that for some  $M \in \text{supp}(\pi)$ ,  $\chi(\pi)(M) = (m, \bar{\omega}_i)$  with  $m \geq 3$  and  $i \in J_0$ , then  $\pi(M)$  can either be  $(m-1)\Lambda_{n+1} + \Lambda_i + \delta_{\mathbf{m}}$  or  $(m-1)\Lambda_{n+1} + \Lambda_i + \theta + \delta_{\mathbf{m}}$ . Suppose  $\pi_1, \pi_2 \in \Pi$  are such that



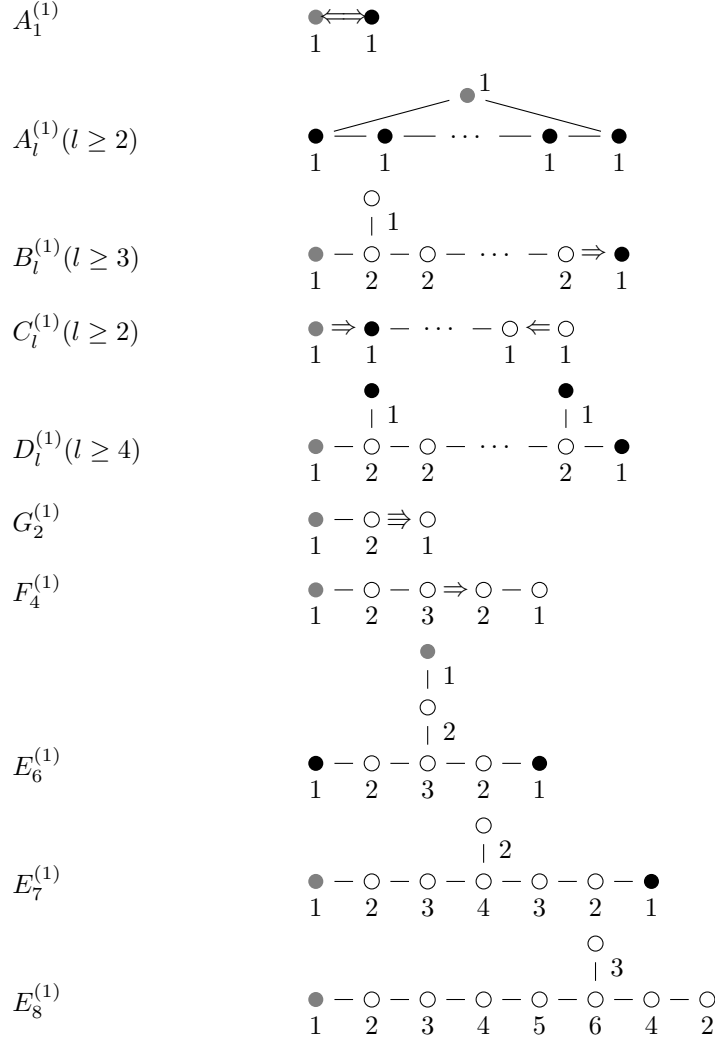


TABLE 1. Dynkin Diagrams of Non-twisted Affine Kac-Moody Lie algebras

In these diagrams the gray nodes correspond to the root  $\alpha_{n+1}$  and the remaining nodes are enumerated as in [Kac, Table Fin]. The numerical labels given here correspond to the number  $a_i^\vee$  (Refer to Section 2.4) and the blackened nodes correspond to those contained in  $J_0$ .

$\pi_1(M) = (m-1)\Lambda_{n+1} + \Lambda_i + \delta_{\mathbf{m}}$  and  $\pi_2(M) = (m-1)\Lambda_{n+1} + \Lambda_i + \theta + \delta_{\mathbf{m}}$  and  $\pi_1(M') = \pi_2(M')$  for all  $M' \in \text{supp}(\pi_1) - \{M\}$ . Then  $\chi(\pi_1) = \chi(\pi_2)$  but  $X_{\pi_1}$  and  $X_{\pi_2}$  are not isomorphic as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules. This gives an example of a function  $\pi \in \Pi$  which is of type **I**. Also from a study of the diagrams above it is easy to see that every  $\pi \in \Pi$  is of type **I** when the associated finite-dimensional Lie algebra  $\mathfrak{g}_{fin}$  is non-simply laced or the corresponding finite group  $\Gamma$  is trivial.

By abuse of nomenclature, we shall say that an element  $\xi \in \Xi$  is of type **I** (respectively **II**) if  $\xi = \chi(\pi)$  for  $\pi \in \Pi$  of type **I** (respectively **II**).

**5.5.** We first consider the finitely supported functions in  $\Xi$  of type **I**.

**Proposition.** *Let  $\xi \in \Xi$  be a finitely supported function of type **I**. Then any two simple  $\mathcal{T}(\mathfrak{g})$ -modules  $V_1, V_2$  in  $\mathcal{I}_{fin}^\xi$  are linked. In particular if  $\pi_1, \pi_2 \in \Pi$  are such that  $\chi(\pi_1) = \chi(\pi_2) = \xi$ , then the irreducible components of  $L(X_{\pi_1})$  are linked to the irreducible components of  $L(X_{\pi_2})$ .*

*Proof.* Let  $\xi \in \Xi$  be a finitely supported function of type **I**. Then there exists  $\pi_1, \pi_2 \in \Pi$  such that  $\chi(\pi_1) = \chi(\pi_2)$  but  $X_{\pi_1}$  is not isomorphic to  $X_{\pi_2}$  as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules. Hence there exists  $M \in \text{supp}(\pi_1)$  such that  $X(\pi_1(M), M)$  is not isomorphic to  $X(\pi_2(M), M)$  as  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$ -modules. Since  $\chi(\pi_1)(M) = \chi(\pi_2)(M)$ , without loss of generality we may assume that  $\pi_1(M) \succ \pi_2(M)$ . That is, there exists  $\beta \in Q_{fin}^+$  and a simple root  $\alpha_i$  of  $(\mathfrak{g}_{fin}, \mathfrak{h}_{fin})$  such that  $(\pi_1(M) - \pi_2(M))|_{\mathfrak{h}_{fin}} = \beta$  and  $\beta(\alpha_i^\vee) > 0$ .

Let  $M = M_1, \dots, M_r$  be an enumeration of the maximal ideals in  $\text{supp}(\pi_2)$  and let  $\pi_3 \in \Pi$  be the finitely supported function with  $\text{supp}(\pi_3) = \text{supp}(\pi_2)$  and

$$\pi_3(M_i) = \begin{cases} \pi_1(M_1) & \text{for } i = 1, \\ \pi_2(M_i), & \text{if } 2 \leq i \leq r. \end{cases}$$

Then  $\chi(\pi_1) = \chi(\pi_3) = \chi(\pi_2)$ . If  $G_{\pi_2}$  is a proper subgroup of  $\mathbb{Z}^{k-1}$  of rank  $k-1$  then for all  $\underline{m} \in \mathbb{Z}^{k-1} - G_{\pi_2}$  (Refer to Equation (4.1) in Section 4.1),

$$\sum_{i=1}^r \pi_2(M_i)(\alpha_i^\vee) \text{ev}_{M_i}(t^{\underline{m}}) = 0.$$

As  $\beta(\alpha_i^\vee) > 0$  and for all  $\underline{r} \in \mathbb{Z}^{k-1}$ ,

$$\sum_{i=1}^r \pi_3(M_i)(\alpha_i^\vee) \text{ev}_{M_i}(t^{\underline{r}}) = \sum_{i=1}^r \pi_2(M_i)(\alpha_i^\vee) \text{ev}_{M_i}(t^{\underline{r}}) + \beta(\alpha_i^\vee) \text{ev}_{M_1}(t^{\underline{r}}),$$

it follows that

$$\sum_{i=1}^r \pi_3(M_i)(\alpha_i^\vee) \text{ev}_{M_i}(t^{\underline{m}}) = \sum_{i=1}^r \pi(M_i)(\alpha_i^\vee) \text{ev}_{M_i}(t^{\underline{m}}) + \beta(\alpha_i^\vee) \text{ev}_{M_1}(t^{\underline{m}}) \neq 0, \quad \forall \underline{m} \in \mathbb{Z}^{k-1} - G_{\pi_2}.$$

Hence  $G_{\pi_3} = \mathbb{Z}^{k-1}$  and consequently  $L(X_{\pi_3})$  is an irreducible  $\mathcal{T}(\mathfrak{g})$ -module. Since tensoring by  $\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  preserves short exact sequences and  $L(X_{\pi_i})$  for  $(i=1,2)$  is completely reducible (by Proposition 4.1) it follows that  $L(X_{\pi_3})$  is linked to each irreducible component of  $L(X_{\pi_i})$  (for  $i = 1, 2$ ). This completes the proof of the proposition in this case.  $\square$

**5.6.** We now consider the finitely supported functions in  $\Xi$  which are of type **II**.

**Proposition.** *Let  $\xi \in \Xi$  be a finitely supported function of type **II**. Then two irreducible  $\mathcal{T}(\mathfrak{g})$ -modules  $X_{\pi_1}^{\mathbf{g}_1}$  and  $X_{\pi_2}^{\mathbf{g}_2}$  in  $\mathcal{I}_{fin}^\xi$  are linked if and only if  $X_{\pi_1}^{\mathbf{g}_1}$  is isomorphic to  $X_{\pi_2}^{\mathbf{g}_2}$  as  $\mathcal{T}(\mathfrak{g})$ -modules.*

*Proof.* Let  $\xi \in \Xi$  be a finitely supported function of type **II**. Given  $\pi, \pi' \in \Pi$  with  $\chi(\pi) = \chi(\pi') = \xi$ , by definition  $X_\pi$  is isomorphic to  $X_{\pi'}$  as  $\mathfrak{g}'_{aff} \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  and  $G_\pi = G_{\pi'}$ . If  $\mathbf{g} \equiv \mathbf{g}' \pmod{G_\pi}$  then by Proposition 4.2,  $X_\pi^{\mathbf{g}}$  is isomorphic to  $X_{\pi'}^{\mathbf{g}'}$  as  $\mathcal{T}(\mathfrak{g})$ -modules and hence they are linked in  $\mathcal{I}_{fin}$ .

Now suppose that  $\mathbf{g}, \mathbf{g}' \in \mathbb{Z}^{k-1}$  are such that  $\mathbf{g} - \mathbf{g}' \notin G_\pi$  and suppose that there exists an indecomposable  $\mathcal{T}(\mathfrak{g})$ -module  $V$  such that

$$X_{\pi'}^{\mathbf{g}'} \xrightarrow{i} V \xrightarrow{p} X_\pi^{\mathbf{g}} \rightarrow 0, \quad (5.3)$$

is a non-split short exact sequence in  $\mathcal{I}_{fin}$ . Then there exists a non-zero weight vector  $v \in V$  such that  $p(v) = v_\pi \otimes t^{\mathbf{g}}$ . Consequently,

$$\begin{aligned} \mathfrak{n}_{aff}^+ \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v &= 0, & h.v &= \sum_{M \in \text{supp}(\pi)} \pi(M)(h).v, \quad \forall h \in \mathfrak{h}_{aff}, \\ K_i t^{\mathbf{m}}.v &= 0, \quad \forall i \geq 2, \mathbf{m} \in \mathbb{Z}^k, & h \otimes t^{\mathbf{m}}.v &= 0, \quad \forall \mathbf{m} \in \mathbb{Z}^{k-1} - G_\pi. \end{aligned}$$

This implies that  $\mathcal{U}(\mathcal{T}(\mathfrak{g})).v$  is a submodule of  $V$  such that  $p(\mathcal{U}(\mathcal{T}(\mathfrak{g})).v) = X_\pi^{\mathbf{g}}$ . Since  $i$  is injective,  $X_\pi^{\mathbf{g}'}$  is irreducible and the sequence (5.3) is non-split it follows that  $\ker p = X_\pi^{\mathbf{g}'}$ , that is,  $X_\pi^{\mathbf{g}'}$  is a submodule of  $\mathcal{U}(\mathcal{T}(\mathfrak{g})).v$ . Hence there must exist an element  $X \in \mathcal{U}(\mathcal{T}(\mathfrak{g}))$  such that  $X.v = v_{\pi'} \otimes t^{\mathbf{g}'}$ . Further using PBW theorem for  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \mathbb{C}d_1$  and the fact that the  $\mathfrak{h}_{aff}$  weight of  $v_{\pi'} \otimes t^{\mathbf{g}'}$  is equal to the  $\mathfrak{h}_{aff}$  weight of  $v$  it follows that the element  $X$  must be in  $\mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}])$ .

Let  $V_0 = \mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]).v$  and for  $\mathbf{r} = (r_2, \dots, r_k) \in \mathbb{Z}^{k-1}$  let

$$V_0[\mathbf{r}] = \{u \in V_0 : d_i.u = r_i u, \ 2 \leq i \leq k\}.$$

Since  $p$  is a  $\mathcal{T}(\mathfrak{g})$ -module homomorphism and  $p(v) = v_\pi \otimes t^{\mathbf{g}}$  it follows that  $V_0[\mathbf{g}] \neq 0$ . On the other hand,

$$h \otimes t^{\mathbf{m}}.v = 0, \quad \forall \mathbf{m} \in \mathbb{Z}^{k-1} - G_\pi.$$

Hence  $V_0[\mathbf{g} + \mathbf{m}] = 0$ , for all  $\mathbf{m} \in \mathbb{Z}^{k-1} - G_\pi$ . In particular  $V_0[\mathbf{g} + (\mathbf{g} - \mathbf{g}')] = 0$  whenever  $\mathbf{g} - \mathbf{g}' \notin G_\pi$  which implies that  $v_\pi \otimes t^{\mathbf{g}'} \notin V_0$  and hence  $X_\pi^{\mathbf{g}'} \cap \mathcal{U}(\mathcal{T}(\mathfrak{g})).v = 0$ . This contradicts our assumption that the sequence is non-split, thereby proving that in this case  $X_\pi^{\mathbf{g}}$  and  $X_\pi^{\mathbf{g}'}$  are not linked unless they are isomorphic that is  $\mathbf{g} \equiv \mathbf{g}' \pmod{G_\pi}$ .  $\square$

Since every module is linked to its submodules and quotients, the following is an immediate consequence of the proposition.

**Corollary.** *Let  $\xi \in \Xi$  be a finitely supported function of type **II**. Then every object in  $\mathcal{I}_{fin}^{\tilde{\xi}}$  is irreducible.*

**5.7.** Finally we are in a position to prove that every indecomposable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  can be uniquely associated with the orbit of a finitely supported function of type **I** or **II** in  $\tilde{\Xi}$ .

**Theorem.** Every indecomposable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}$  is an object of  $\mathcal{I}_{fin}^{\tilde{\xi}}$  for some  $\xi \in \Xi$ .

*Proof.* Let  $V$  be an irreducible  $\mathcal{T}(\mathfrak{g})$ -module. By Proposition 4.1 there exists  $\pi \in \Pi$  such that  $V$  is isomorphic to  $X_\pi^{\mathbf{g}}$  for some  $\mathbf{g} \in \mathbb{Z}^{k-1}$ . Thus in this case  $V \in \mathcal{I}_{fin}^{\chi(\pi)}$ .

Suppose  $V$  in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  is reducible. Then by Proposition 3.3,  $V$  contains an irreducible  $\mathcal{T}(\mathfrak{g})$ -module  $X_\pi^{\mathbf{g}}$ . Since  $X_\pi^{\mathbf{g}}$  is linked to  $V$ , by Corollary 5.6,  $\pi \in \Pi$  is of type **I**. Let  $U$  be the corresponding quotient of  $V$  such that is we have the extension

$$0 \rightarrow X_\pi^{\mathbf{g}} \rightarrow V \rightarrow U \rightarrow 0.$$

By Lemma 2.5(i),  $X_\pi^{\mathbf{g}}, U \in \text{Ob } \mathcal{I}_{fin}^{(me_1)}$  and hence by Theorem 3.2,  $U$  can be written as the direct sum of indecomposable  $\mathcal{T}(\mathfrak{g})$ -modules  $U_j, j = 1, \dots, r$ . Clearly the length of each  $\mathcal{T}(\mathfrak{g})$ -module  $U_j$  is strictly less than the length of  $V$ . Hence by inductive hypothesis there exists  $\xi_j \in \Xi$  such that  $U_j \in \text{Ob } \mathcal{I}_{fin}^{\xi_j}$  for  $j = 1, \dots, r$ . Suppose that  $\xi_j \neq \chi(\pi)$  for some  $1 \leq j \leq r$ . Then by Corollary 5.3(i),  $\text{Ext}_{\mathcal{T}(\mathfrak{g})}^1(U_j, X_\pi^{\mathbf{g}}) = 0$ , which implies that there exists a direct summand of  $V$  that is isomorphic to  $U_j$ . This contradicts our assumption that  $V$  is indecomposable. Hence in this case  $U_j \in \mathcal{I}_{fin}^{\chi(\pi)}$  for all  $1 \leq j \leq r$ .  $\square$

## REFERENCES

- [B] N. Bourbaki, *Elements of Mathematics, Lie Groups and Li Algebras*, Chapters 4-6.
- [C] V. Chari, *Integrable representations of affine Lie-algebras*, Invent. Math. **85** (1986), 317-335.
- [CFK] V. Chari, G. Fourier, T. Khandai, *A Categorical Approach to Weyl Modules*, Transformation Groups **15**(3), (2010), 517-549.
- [CG] V. Chari and J. Greenstein, *Graded Level Zero Integrable Representations of Affine Lie Algebras*, Transactions of AMS. **360** (2008), no. 6, 2923-2940.
- [CM] V. Chari, A. Moura, *Spectral Characters of finite-dimensional representations of affine algebras*, J. Algebra **279** (2004), no.2, 820-839.
- [CP1] V. Chari, A. Pressley, *New unitary representations of loop algebras*, Math. Ann. **275**, (1986), no.1 87-104.
- [CP2] V. Chari, A. Pressley, *A new family of Irreducible, Integrable Modules for Affine Lie algebras*, Math. Ann. **277**, (1987), 543-562.
- [CL] V. Chari, T. Le, *Representations of Double Affine Lie Algebras*, In a Tribute to C. S. Seshadri (Chennai, 2002) Trends Math., Birkhauser, Basel, 2003, 199-219.
- [FL] G. Fourier, P. Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions*, Adv. Math. **211**(2) (2007), 566-593.
- [H] J.E.Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1968
- [K] C. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebras*, Journal of Pure and Applied Algebra **34** (1985), 256-275
- [Kac] V. G. Kac, *Infinite dimensional Lie algebras* Prog. Math., Boston 44, Third Edition.
- [Kh] T. Khandai, *Irreducible Integrable Representations of Toroidal Lie Algebras*, arXiv:1607.04389
- [KL] D. Kus, P. Littelmann, *Fusion products and toroidal algebras*, Pacific J. Math. **278** (2015) 427-445.
- [Ko] R. Kodera, *Extensions between finite-dimensional simple modules over a generalized current Lie algebra*, Transform. Groups **15** (2010), no. 2, 371-388.
- [NS] E. Neher, A. Savage, *Extensions and block decompositions for finite-dimensional representations of equivariant map algebras*, Transform. Groups **20**, (2015), no. 1, 183-228.
- [S] P. Senesi, *The Block Decomposition of finite-dimensional representations of twisted loop algebras*, Pacific J. Math. **244** No. 2, (2010) 335-357.
- [R1] S. E. Rao, *Complete reducibility of integrable modules for the affine Lie (super)algebras*, Journal of Algebra **264** (2003), no. 1, 269-278.
- [R2] —, *Classification of irreducible integrable modules for toroidal Lie algebras with finite dimensional weight spaces*, Journal of Algebra **277** (2004), 318-348.
- [R3] —, *Irreducible representations for toroidal Lie-algebras*, Journal of Pure and Applied Algebra **202**(1-3) (2005), 102-117.
- [RM] S. E. Rao, R.V. Moody, *Vertex Representations for N-Toroidal Lie algebras and a Generalization of the Virasoro Algebras*, Communications in Mathematical Physics **159** (1994), 239-264.
- [RSF] S. E. Rao, S. Sharma, V. Futorny, *Weyl Modules associated to Kac-Moody Lie Algebras*, Comm. Algebra **44** (2016), no. 12, 5045–5057 (arXiv:1501.04802).
- [VV] R. Venkatesh, S. Viswanath, *Unique factorization of tensor products for Kac-Moody algebras*. Adv. Math. **231** (2012), no. 6, 3162–3171

NATIONAL INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, KHURDA 752050, INDIA

E-mail address: tanusree@niser.ac.in